The universal Kobayashi-Hitchin correspondence on Hermitian manifolds

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Abstract

We prove a very general Kobayashi-Hitchin correspondence on arbitrary compact Hermitian manifolds, and we discuss differential geometric properties of the corresponding moduli spaces. This correspondence refers to moduli spaces of "universal holomorphic oriented pairs". Most of the classical moduli problems in complex geometry (e. g. holomorphic bundles with reductive structure groups, holomorphic pairs, holomorphic Higgs pairs, Witten triples, arbitrary quiver moduli problems) are special cases of this universal classification problem. Our Kobayashi-Hitchin correspondence relates the complex geometric concept "polystable oriented holomorphic pair" to the existence of a reduction solving a generalized Hermitian-Einstein equation. The proof is based on the Uhlenbeck-Yau continuity method. Using idea from Donaldson theory, we further introduce and investigate canonical Hermitian metrics on such moduli spaces. We discuss in detail remarkable classes of moduli spaces in the non-Kählerian framework: Oriented holomorphic structures, Quot-spaces, oriented holomorphic pairs and oriented vortices, non-abelian Seiberg-Witten monopoles.

0   Introduction

In order to understand the aim and the motivation of this article let us consider the following *classical complex geometric moduli problems*:

Holomorphic structures with fixed determinant.   Let X be a compact complex n-dimensional manifold and E a differentiable rank r vector bundle on X. We fix a holomorphic structure $\mathcal{L}$ on the determinant line bundle $\det E$ of X. The problem is to classify all holomorphic structures $\mathcal{E}$ on E which induce the fixed holomorphic structure $\mathcal{L}$ on $\det E$ modulo the group $\Gamma(X, SL(E))$ of automorphisms of determinant 1.

In order to get a Hausdorff moduli spaces with good properties one considers only *stable* (or more general *semistable*) holomorphic structures. The stability condition depends on the choice of a Gauduchon metric on X, or more generally a Hermitian metric g (see [LT]) which plays the same role as the choice of a polarization of the base manifold in algebraic geometry. $\mathcal{E}$ is called g-stable if

$$\mu_g(\mathcal{F}) < \mu_g(\mathcal{E}) \tag{1}$$

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for any nontrivial subsheaf $\mathcal{F} \subset \mathcal{E}$ with torsion free quotient. Such a subsheaf must be reflexive. Note that the slope map μ_g is <u>not</u> a topological invariant in the general non-Kählerian framework, but it is always a holomorphic invariant.

The *classical Kobayashi-Hitchin correspondence* ([Ko1], [Do1], [Do2], [UY1], [UY2], [Bu], [LY], [LT]) states that a holomorphic structure is stable if and only if it is simple (i. e. it admits no non-trivial trace free infinitesimal automorphisms) and admits a Hermitian-Einstein metric, i. e. a metric h solving the Hermitian-Einstein equation:

$$ i\Lambda_g F_h = \frac{2\pi}{(n-1)! Vol_g(X)} \mu_g(\mathcal{E}) \mathrm{id}_E \ . $$

General solutions of the Hermitian-Einstein equation correspond to *polystable* holomorphic structures, i. e. to bundles which are direct sums of stable bundles of the same slope.

From a historical point of view, around 1980 the correspondence was independently suggested by Hitchin and Kobayashi (according to Donaldson [Do1] who first linked the two names together[1]). First indications that a connection between these two concepts might exist had been e.g. Kobayashi's proof that a Hermitian-Einstein bundle is stable in the sense of Bogomolov [Ko4], and the first author's proof of the Chern class inequality for Hermitian-Einstein bundles [Lü1]. The "simple" implication in this case, i.e. the polystability of Hermitian-Einstein bundles, was first written down in Japanese by Kobayashi in [Ko3] and announced in [Ko1]; the first author then found a simpler proof [Lü2]. The "difficult" implication, i.e. the existence of a Hermitian-Einstein connection in a stable bundle, was subsequently proved by Donaldson for Riemann surfaces [Do3], algebraic surfaces [Do1] and manifolds [Do2], by Uhlenbeck and Yau for Kähler manifolds [UY1], [UY2], by Buchdahl for arbitrary Hermitian surfaces [Bu], and finally by Li and Yau for arbitrary Hermitian manifolds [LY]. In the last two articles, as in the present work, an important technical tool is the use of a *Gauduchon* metric in the base manifold. An extensive discussion of the classical correspondence, including complete proofs and several applications, can be found in [LT].

Let us fix a Hermitian metric h on E. The Kobayashi-Hitchin correspondence yields an isomorphism of moduli spaces between the moduli space of stable holomorphic structures with fixed determinant and the moduli space of irreducible integrable Hermitian-Einstein connections A with fixed determinant on (E, h). This isomorphism plays a fundamental role in Donaldson theory, since – on complex surfaces – the latter moduli space can be further identified with a moduli space of projectively ASD connections (see [DK], [LT] and section 5.1.2 in this article for details).

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[1]We thank the referee for pointing out this fact and reference [KO].

Higgs pairs. We fix a differentiable vector bundle E on X, a holomorphic structure $\mathcal{L}$ on $\det E$ and a holomorphic vector bundle $\mathcal{F}_0$ on X. In the classical theory of Higgs fields one takes $\mathcal{F}_0 = \Omega_X^1$ (see [Hi], [Si]).

The problem here is to classify modulo $\Gamma(X, SL(E))$ all pairs $(\mathcal{E}, \Phi)$ where $\mathcal{E}$ is again a holomorphic structure on E which induces $\mathcal{L}$ on $\det E$, and $\Phi \in H^0(\mathrm{End}(\mathcal{E}) \otimes \mathcal{F}_0)$ is a holomorphic $\mathcal{F}_0$-twisted endomorphism.

For such objects one also has a stability condition: this condition asks that (1) holds for all nontrivial Φ-invariant subsheaves $\mathcal{F}$. This complex geometric condition can be again characterized in a differential geometric way (see [Hi], [Si]): a Higgs holomorphic pair $(\mathcal{E}, \Phi)$ is stable if and only if is simple and $\mathcal{E}$ admits a metric h satisfying the Higgs equation

$$i\Lambda F_h + \frac{1}{2}[\Phi, \Phi^{*h}] = \frac{2\pi}{(n-1)! Vol_g(X)} \mu_g(\mathcal{E}) \mathrm{id}_E$$

Again, this result yields an isomorphism of moduli spaces: the complex geometric moduli space of stable holomorphic Higgs pairs and the differentiable geometric moduli space of irreducible integrable solutions (A, Φ) of the Higgs equation. Arbitrary solutions of this equation correspond to polystable holomorphic Higgs pairs (see [Hi], [Si]).

Holomorphic pairs. (see [Bra], [Th], [HL]) Let E be a rank r bundle on X. One wants to classify – modulo $\Gamma(X, GL(E))$ – the pairs $(\mathcal{E}, \varphi)$, where $\mathcal{E}$ is a holomorphic structure on E and φ is a holomorphic section in $\mathcal{E}$.

The stability condition for this problem depends on a real parameter τ. A pair $(\mathcal{E}, \varphi)$ is τ-stable if

$$\max(\mu_g(\mathcal{E}), \sup_{\mathcal{F} \in \mathcal{R}(E)} \mu_g(\mathcal{F})) < \tau < \inf_{\substack{\mathcal{F} \in \mathcal{R}(E) \\ \varphi \in H^0(\mathcal{F})}} \mu_g(\mathcal{E}/\mathcal{F}) \ ,$$

where $\mathcal{R}(\mathcal{E})$ denotes the set of subsheaves of $\mathcal{E}$ with torsion free quotients.

There is again a differential geometric characterization of the stable pairs. $(\mathcal{E}, \varphi)$ is τ-stable if and only if it is simple (i. e. it admits no non-trivial infinitesimal automorphisms) and $\mathcal{E}$ admits a metric h satisfying the *vortex equation*
$$i\Lambda F_h + \frac{1}{2}\varphi \otimes \varphi^{*h} = \frac{2\pi}{(n-1)! Vol_g(X)} \ \tau \mathrm{id}_E \ .$$

Witten triples. (see [W], [Dü]) Consider a line bundle L on a complex surface X. We want to classify modulo $\Gamma(X, GL(L)) = \mathcal{C}^\infty(X, \mathbb{C}^*)$ the triples $(\mathcal{L}, \varphi, \alpha)$

consisting of a holomorphic structure $\mathcal{L}$ on L, a holomorphic section $\varphi \in H^0(\mathcal{L})$ and a holomorphic morphism $\alpha : \mathcal{L} \to \mathcal{K}_X$.

The stability condition depends again on a real parameter τ. A triple $(\mathcal{L}, \varphi, \alpha)$ is τ-stable if either $\deg(\mathcal{L}) < \tau$ and $\varphi \neq 0$, or $\deg(\mathcal{L}) > \tau$ and $\alpha \neq 0$.

A triple $(\mathcal{L}, \varphi, \alpha)$ is stable if and only if it is simple (i. e. $(\varphi, \alpha) \neq 0$) and $\mathcal{L}$ admits a metric h satisfying the *mixed vortex equation*

$$ i\Lambda_g F_h + \frac{1}{2}(|\varphi|^2 - |\alpha|^2) = \frac{2\pi}{Vol_g(X)} \ \tau \ . $$

This result yields again an isomorphism of moduli spaces.

Remark: In the vortex equation and the mixed vortex equation one can replace the constant $\frac{2\pi}{(n-1)!Vol_g(X)} \tau$ on the right by a real function t. The corresponding stability condition depends only on $\int_X t \ vol_g$ when g is Gauduchon. The case when $t = \frac{s_g}{2}$ (where s_g stands for the scalar curvature of g) plays a crucial role in the Seiberg-Witten theory on complex surfaces (see [W], [Bi], [Dü] [OT1], [OT2], [Te2]).

Oriented holomorphic pairs. We fix again a differentiable vector bundle E of rank r on X and a holomorphic structure $\mathcal{L}$ on $\det E$. This time we want to classify modulo $\Gamma(X, SL(E))$ the pairs $(\mathcal{E}, \varphi)$, where $\mathcal{E}$ is a holomorphic structure on E which induces $\mathcal{L}$ on $\det E$ and $\varphi \in H^0(\mathcal{E})$ is a holomorphic section.

The stability condition in this case is (see [OST], [OT2], [OT8], [Te2])

$$ \max(\mu_g(\mathcal{E}), \sup_{\mathcal{F} \in \mathcal{R}(E)} \mu_g(\mathcal{F})) < \inf_{\substack{\mathcal{F} \in \mathcal{R}(E) \\ \varphi \in H^0(\mathcal{F})}} \mu_g(\mathcal{E}/\mathcal{F}) \ , $$

whereas the corresponding Hermitian-Einstein type equation is

$$ i\Lambda F_h + \frac{1}{2}(\varphi \otimes \varphi^{*h})_0 = 0 $$

In the case $n = r = 2$, one gets a complex geometric interpretation of the moduli spaces of $PU(2)$-monopoles on complex surfaces (see [Te2] for the Kählerian case and section 5.3 in this article for the Gauduchon case).

In order to illustrate the generality of the results in this article, consider now the following *artificial moduli problem*:

Fix a system of differentiable vector bundles $(E_i)_{0 \leq i \leq n}$ of ranks r_i, a system of holomorphic vector bundles $(\mathcal{F}_j)_{1 \leq j \leq m}$ of ranks ρ_j on X, and a holomorphic structure $\mathcal{L}_0$ on $\det E_0$.

Classify systems $(\mathcal{E}_i, \varphi_{ij}, \psi_i)$ where $\mathcal{E}_i$ is a holomorphic structure on E_i such that $\mathcal{E}_0$ induces $\mathcal{L}_0$ on $\det E_0$, $\varphi_{ij} \in H^0(\mathcal{E}_i^\vee \otimes \mathcal{F}_j)$ are holomorphic homomorphisms, and $\psi_i \in \Gamma(X, \mathbb{P}(\mathrm{End}(\mathcal{E}_i)))$ are holomorphic sections in the projectivizations of the endomorphism bundles of $\mathcal{E}_i$.

The question is: which is the correct stability (polystability) condition in this case, and which is the corresponding Hermitian-Einstein equation?

All the "classical" moduli problems listed above, as well as this artificial moduli problem are just special case of the following very general complex geometric classification problem:

A universal moduli problem.   We note that in all these examples one has a system of *fixed holomorphic structures*, a system of *variable holomorphic structures* on fixed $\mathcal{C}^\infty$-bundles, and a system of *variable sections* in associated bundles, which are holomorphic with respect to both variable and fixed holomorphic structures.

The best way to formulate a universal generalization of all moduli problems of this type is the following (see [OT3], [OT5]):

Consider an exact sequence

$$\{1\} \longrightarrow G \longrightarrow \hat{G} \longrightarrow G_0 \longrightarrow \{1\}$$

of complex reductive groups, and choose a $\hat{G}$-bundle $\hat{Q}$ on X and a holomorphic action $\hat{\alpha} : \hat{G} \times F \to F$ on a Kählerian manifold F. Let $Q_0 := \hat{Q}/G$ be the associated G_0-bundle and E the associated F-bundle $E := \hat{Q} \times_{\hat{\alpha}} F$. Fix a holomorphic structure $\mathcal{Q}_0$ on Q_0. Our universal classification problem is:

Classify the pairs $(\hat{\mathcal{Q}}, \varphi)$, where $\hat{\mathcal{Q}}$ is a holomorphic structure on $\hat{Q}$ inducing $\mathcal{Q}_0$ on Q_0 and $\varphi \in \Gamma(X, E)$ is holomorphic with respect to the holomorphic structure induced by $\hat{\mathcal{Q}}$ on E, modulo the gauge group $\mathcal{G} := \mathrm{Aut}_{Q_0}(\hat{Q}) = \Gamma(X, \hat{Q} \times_{\mathrm{Ad}} G)$.

A pair $(\hat{\mathcal{Q}}, \varphi)$ of this type will be called an *oriented pair of type* $(\hat{Q}, \hat{\alpha}, \mathcal{Q}_0)$.

This class of moduli problems was first considered in [OT3] in the case when X is a complex curve; in this case the corresponding moduli spaces were used to introduce the *twisted gauge theoretical Gromov-Witten invariants*. The particular case $G_0 = \{1\}$ (the case of *non-oriented pairs*) was previously studied by Mundet i Riera in [Mu1] on Kähler manifolds.

Example: For instance, to get our "artificial moduli problem" above as a

special case of our universal problem, take

$$G := SL(r_0, \mathbb{C}) \times \prod_{i=1}^{n} GL(r_i, \mathbb{C}) \,, \ \hat{G} := GL(r_0, \mathbb{C}) \times \prod_{i=1}^{n} GL(r_i, \mathbb{C}) \times \prod_{j=1}^{m} GL(\rho_j, \mathbb{C}) \,,$$

$$G_0 = \mathbb{C}^* \times \prod_{j=1}^{m} GL(\rho_j, \mathbb{C}) \,, \ F := \prod_{\substack{0 \leq i \leq n \\ 1 \leq j \leq m}} \mathrm{Hom}(\mathbb{C}^{r_i}, \mathbb{C}^{\rho_j}) \times \prod_{i=0}^{n} \mathbb{P}(\mathrm{End}(\mathbb{C}^{r_i})) \,.$$

The action $\hat{\alpha}$ is the natural action of $\hat{G}$ on F, whereas the fixed holomorphic structure $\mathcal{Q}_0$ is defined by the system of fixed holomorphic structures $(\mathcal{L}, (\mathcal{F}_j)_{1 \leq j \leq m})$.

Similarly one can see easily that all classical moduli problems in our list are also special cases of this universal problem, as well as all "quiver problems" considered in [AG].

The main goals of our article are:

I. To give the correct stability and polystability condition for this universal moduli problem and to characterize the polystable pairs in terms of the existence of a reduction of $\hat{\mathcal{Q}}$ to a maximal compact subgroup $\hat{K} \subset \hat{G}$ which satisfies a certain generalized Hermitian-Einstein equation. In this article, we solve this problem completely (see Theorem 3.3, Theorem 4.2 for the precise statements).

In particular, these results give a Kobayashi-Hitchin correspondence for holomorphic principal bundles with arbitrary reductive groups on arbitrary compact Hermitian manifolds.

II. To study, in the non-Kählerian framework, metric properties of this large class of moduli spaces: we show that the (smooth part of the) moduli space associated with such a problem comes always with a natural Hermitian metric, and that this Hermitian metric has interesting properties (for instance is Kähler as soon as the base manifold is semi-Kähler). The construction of this *canonical Hermitian metric* uses ideas from Donaldson theory.

II. To apply our general results to remarkable classes of moduli spaces: moduli spaces of oriented connections, Douady Quot-spaces, moduli spaces of non-abelian monopoles on Gauduchon surfaces.

Previous results on universal moduli problems. In this paragraph we acknowledge previous contributions on the subject, from which we benefited a lot. On the other hand we explain carefully the progress made in the present article compared with previous work.

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The idea to a find a "universal" generalization of the correspondence for *all possible coupled* moduli problems is of course very natural. Important progress in this direction was first achieved by Banfield [Ba] who proved a Kobayashi-Hitchin correspondence for *non-oriented* pairs whose second component is a section in vector bundle associated with a *linear representation* α. Banfield's stability condition is a natural extension to coupled problems of the stability condition for holomorphic bundles with arbitrary reductive structure group as developed by Ramanathan and Ramanan-Ramanathan [Ra], [RR]. In his very interesting Ph. D. thesis (see [Mu1]), Mundet i Riera succeeded to generalize Banfield's result to the case of an arbitrary (not necessary linear) action on a Kählerian fibre. In the present article we use essentially his important contribution on the subject.

We also mention that a very general Kobayashi-Hitchin correspondence on Kähler manifolds is stated in [BGM], where the same type of holomorphic pairs is considered, but the symmetry group of the problem is a more general "Lie subgroup" $\mathcal{G} \subset \mathrm{Aut}(\hat{Q})$; but there is a price to pay for this generality: the inequality in the corresponding stability condition depend on a vector varying in an infinite dimensional Lie algebra, so this is not a practically checkable condition.

All these contributions concern the particular case when the base manifold is Kählerian and they all use similar methods.

Compared with the existing literature, the results and the formalism developed in the present article has several important advantages :

1. Our methods applies to the general non-Kählerian framework.

    We mention that in the non-Kählerian framework several important difficulties arise:

    (a) The left term of the Hermitian-Einstein equation can no longer be regarded as a moment map,

    (b) The Kähler identities do not hold,

    (c) The degree of holomorphic bundles is no longer a topological invariant.

    For these reasons, the methods used in [Mu1], [BGM] do not appear to apply to the non-Kählerian framework.

2. We treat not only the correspondence between stable objects and irreducible solutions of the (generalized) Hermitian-Einstein equations (see [Mu1], [BGM]) but the general correspondence between *polystable* objects and *arbitrary* solutions of these equations.

This is also an important point, because even in the classical Donaldson theory, the possible presence of the "reductions" in the moduli spaces must be always carefully taken into account.

3. Our stability condition has *holomorphic character*; more precisely, the terms in our stability inequalities are intrinsic *holomorphic invariants* associated with the oriented holomorphic pair $(\hat{\mathcal{Q}}, \varphi)$ and certain *meromorphic reductions* of $\hat{\mathcal{Q}}$. Only the *given* parameters (the Hermitian base manifold and the Hamiltonian action on the fibre) are used in the construction of these invariants. *Holomorphically isomorphic oriented pairs will have the same set of invariants.*

   These invariants *do not require and do not depend on a background $\hat{K}$ - reduction* of $\hat{\mathcal{Q}}$ (contrary to [Mu1] and [BGM]). This is important (even in the Kähler and algebraic case!), because a $\hat{K}$-reduction is a differential geometric object; its construction requires a partition of unity (as does the construction of a Hermitian structure of a vector bundle), so a stability condition which depends on the choice of a $\hat{K}$-reduction is not practically checkable.

   The complex geometric character of our stability condition is a consequence of the complex geometric equivariance properties – obtained in [Te3] – of the maximal weight function associated with a finitely dimensional *energy complete Hamiltonian triple* (see Proposition 1.9). Without the energy completeness condition, there seems to be no way to prove these complex geometric equivariance properties.

The main feature of the classical stability condition for bundles is its complex geometric nature: even on general Hermitian manifolds the slope map on the set of non-trivial subsheaves of a given holomorphic bundle has a pure complex geometric character, because it does not depend on the choice of a Hermitian metric on this bundle (see [Ga], [LT]).

Therefore our universal Kobayashi-Hitchin correspondence respects this fundamental feature of the classical Kobayashi-Hitchin correspondence, i. e. it deals only with complex geometric stability and polystability conditions.

In fact this difficulty already arises in the finite dimensional framework (see section 1): the generalized maximal weight function λ associated with a K-moment map μ for a general holomorphic action $\alpha : G \times F \to F$ on a Kähler manifold is not equivariant with respect to the whole complex reductive group G, but only with respect to the fixed maximal compact subgroup K. Therefore, the inequality $\lambda(x) > 0$ involved in the "analytic stability condition" is a priori *not* a complex geometric condition. Moreover, in the general case, it is not possible to give numerical characterizations of semistable and polystable points.

In the finite dimensional framework these difficulties have been explained and

overcome in [Te3], where it was shown that one has to impose a certain completeness condition on the pair (α, μ) in order to have a maximal weight function λ with good complex geometric equivariance properties. For this class of actions (which includes all holomorphic Hamiltonian actions on compact Kähler manifolds and all linear actions) one can give complex geometric numerical characterizations of stable, semistable and polystable orbits.

Using this formalism, we were able to extend the Kobayashi-Hitchin correspondence for "universal" oriented holomorphic pairs to the case when the base manifold X is an arbitrary compact Hermitian manifold. Moreover, we give a complex geometric numerical characterization not only of the stable, but also of the polystable pairs (i. e. of all pairs which admit a $\check{K}$-reduction satisfying the generalized Hermitian-Einstein equation). This gives a complete complex geometric interpretation of the whole moduli space of solutions of the generalized Hermitian-Einstein equation.

Extending the Kobayashi-Hitchin correspondence to the non-Kählerian framework is important for many reasons: for instance this generalization furnishes complex geometric descriptions of the moduli spaces of instantons with arbitrary compact symmetry groups and of the moduli spaces of Seiberg-Witten monopoles (abelian and non-abelian) on all complex surfaces. Furthermore, important applications of the classical Kobayashi-Hitchin correspondence to the classification of non-Kählerian surfaces can be found in [Te1], [Te4] [LT], [LYZ].

Our proof is based on the continuity method developed by Uhlenbeck and Yau for the classical Kobayashi-Hitchin correspondence (relating polystable bundles to Hermitian-Einstein connections). This seems to be the appropriate method in the non-Kählerian framework, because in this case the left hand term in the generalized Hermitian-Einstein equation *is no longer a formal moment map*. On the other hand, the original proof of Uhlenbeck-Yau for the classical Kobayashi-Hitchin correspondence does not extend mechanically to our class of coupled problems with general symmetry groups, so the classical strategy developed by these authors must be completed with new techniques.

Usually moduli problems for bundles with reductive structure group G are treated using a faithful representation $G \to GL(r, \mathbb{C})$ and reducing the problem to the case $G = GL(r, \mathbb{C})$, i. e. to the more familiar case of vector bundles. We will *not* follow this tradition, because we realized that this method does not really simplify the problem, and dealing directly with the general case is more natural and interesting. Consequently, we need an analogue of the well-known theorem of Uhlenbeck-Yau on weakly holomorphic subbundles, for general reductive structure groups G. In other words, we show how one can construct a meromorphic reduction of a holomorphic principal G-bundle $\mathcal{Q}$ to a parabolic subgroup $L \subset G$ using an L_1^2-section $s \in \Gamma(\mathrm{ad}(\mathcal{Q}))$ which satisfies an algebraic property and a certain weak holomorphy condition. This correspondence, which follows easily from the results in [UY1], [UY2], is presented in the Appendix.

The structure of the article. We begin with the "finite dimensional" Kobayashi-Hitchin correspondence. This result gives a numerical characterization of the polystable orbits with respect to a holomorphic Hamiltonian action (α, μ) satisfying our technical completeness condition. This result can be regarded as a (*non-algebraic*!) complex geometric generalization of the well-known Hilbert criterion in classical GIT. A different proof can be found in [Mu1] (for the stable case) and [Te3] for the general semistable case. Here we give a proof which is based on the same continuity method which will be used later in the infinite dimensional gauge theoretical framework.

The main ideas of the results presented here are present already in the classical GIT contributions ([Ki], [MFK], [KN]); the goal of this chapter is just to treat carefully the delicate *non-algebraic non-compact* case and to point out the specific difficulties (see [Te3], [BT]).

In the second chapter we state our "universal" moduli problem and we introduce the stability, semistability and polystability conditions for universal oriented pairs. Much care is dedicated here to the notion of *degree* of a meromorphic reduction to a parabolic subgroup with respect to an ad-invariant linear form; this concept is very delicate in the non-Kähler framework.

In the third chapter we introduce the Hermitian-Einstein equation for universal oriented pairs and we prove the "simple" implication of the Kobayashi-Hitchin correspondence: any pair which admits a Hermitian-Einstein reduction is polystable.

The fourth chapter deals with the "difficult" implication: any polystable pair admits an admissible Hermitian-Einstein reduction. The proof of this, based on the Uhlenbeck-Yau continuity method, uses involved analytical techniques.

In the fifth chapter we give applications of our results: The starting point of the chapter is to notice that, as in Donaldson theory, the obtained correspondence between solutions of the generalized Hermitian-Einstein equation and polystable oriented pairs yields always an isomorphism of moduli spaces (the Kobayashi-Hitchin isomorphism). In the first section of the chapter we treat in detail the case of the correspondence between oriented holomorphic structures and oriented connections. We give a simple argument to explain why moduli spaces of *oriented* connections are important: the "cobordism argument" used for relating the Donaldson invariants to the Seiberg-Witten invariants generalizes to non-simply connected 4-manifolds, but one should use Donaldson invariants constructed with moduli spaces of oriented, projectively ASD $U(2)$-connections, instead of moduli spaces of $PU(2)$-instantons. In the second section of the chapter we turn to coupled problems. For both classes of moduli problems (non-coupled and coupled) we use the Kobayashi-Hitchin isomorphism to construct a canonical Hermitian metric on (the smooth part of) the moduli space. We prove in full generality that this metric is always Kähler when the base man-

ifold is semi-Kähler, and that its Kähler form Ω satisfies the equation $\partial\bar{\partial}\Omega = 0$, when the base manifold is Gauduchon. The main idea of the proof comes from Donaldson theory: for the construction of Ω we use a method similar to the one used in [DK] to construct de Rham representatives of the μ-classes. This generalizes the results concerning the metric properties of the classical moduli spaces of holomorphic vector bundles, obtained with different methods in [LT]. As an application we endow Douady Quot-spaces with canonical Hermitian metrics, by identifying them with suitable moduli spaces of stable oriented holomorphic pairs. The more delicate non-Kählerian case is treated carefully. The third section of the chapter is dedicated to non-abelian Seiberg-Witten theory for Gauduchon surfaces. We give a geometric interpretation of the moduli spaces of non-abelian monopoles in this general framework in terms of oriented rank 2 holomorphic pairs.

The Appendix contains technical results (i.e. identities for Chern connections, local diagonalization results for arbitrary compact Lie groups, meromorphic reductions to parabolic subgroups) which are all of independent interest.

Notations and conventions. We tried to respect the standard conventions and notations used in the literature. However, for mathematical reasons, we also had to introduce some new ones:

For instance, in [Mu1] the notations E, E_G are used for a principal K bundle and its $G = K^{\mathbb{C}}$-extension $E \times_K G$. In our article, $\hat{P}$ stands for a $\hat{K}$ principal bundle (standard differential geometric notation) and $\hat{Q}$ for a $\hat{G}$-principal bundle, where $\hat{K}$ is a maximal compact subgroup in $\hat{G}$. It is very important that we do *not* fix an identification $\hat{Q} = \hat{P} \times_{\hat{K}} \hat{G}$ when stability is introduced, because fixing such an identification is equivalent to choosing a background $\hat{K}$-reduction of $\hat{Q}$, and, as explained above, *stability should be a holomorphic condition referring to holomorphic objects.*

For the same reason, the first component of our holomorphic pairs is a bundle-holomorphic structure on $\hat{Q}$ (denoted by $\hat{J}$ or $\hat{\mathcal{Q}}$) and not an integrable $\hat{K}$-connection $\hat{A}$ (compare with [Mu1], [BGM]). Passing from holomorphic structures to connections is possible only after fixing a $\hat{K}$-reduction of $\hat{Q}$.

Third, the real and complex gauge groups are denoted by $\mathscr{G}_K$ and $\mathscr{G}_G$ in [Mu1]. To simplify the notation, we preferred the symbols $\mathcal{K}$ and $\mathcal{G}$.

Note also that our stability condition is slightly simpler than the stability condition in [Mu1]. Indeed, the "initial parameter" in [Mu1] is a pair (L, ξ) consisting of a parabolic subgroup of G and a dominant character of L. On the other hand, our initial parameter is just a Hermitian type vector ξ in the Lie algebra $\mathfrak{g}$. The point is that we only need a parabolic subgroup of G which is *canonically associated with ξ.* Since the functoriality of this parabolic group

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with respect to both G and ξ plays an important role, we used the notation $G(\xi)$ for this parabolic subgroup.

Moreover, the central constant c in the "c-stability condition" formulated in [Mu1] can be absorbed in the moment map μ, so without any loss of generality, one can omit this parameter.

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1 The finite dimensional Kobayashi-Hitchin correspondence

In this section we give a brief presentation of the stability theory in the finitely dimensional Kählerian *non-algebraic* framework and explain the analogue of the Hilbert criterion in this framework. The Hilbert criterion we prove here gives a numerical characterization of the polystable orbits, i. e. of the orbits which intersect the vanishing locus of the moment map. Our proof – based on the continuity method – is a very good preparation for understanding the proof of our universal Kobayashi-Hitchin correspondence on Hermitian manifolds in section 4, which will follow the same strategy but will require involved analytical techniques.

1.1 Analytic Stability, Symplectic stability

For complete proofs and more details about the notions and the results introduced in this section we refer to [Te3]. The analogous results in the algebraic geometric framework are well known ([Ki], [MFK], [KN]), but the non-algebraic framework raises specific difficulties.

Let G be a complex reductive group. We denote by $H(G)$ the subset $H(G) \subset \mathfrak{g}$ defined by

$$H(G) := \{\xi \in \mathfrak{g}|\ \overline{\exp(i\mathbb{R}\xi)} \text{ is compact}\} = \bigcup_{\substack{K \subset G \\ \text{maximal compact}}} i\mathfrak{k} \ .$$

$H(G)$ is a locally closed cone in $\mathfrak{g}$. The elements in $H(G)$ will be called vectors of Hermitian type (see [Te3]). There is an obvious *non-surjective* injective map

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$\mathrm{Hom}(\mathbb{C}^*, G) \to H(G)$ given by

$$\theta \mapsto \left.\frac{d}{dt}\right|_{t=0} (\mathbb{R} \ni t \mapsto \theta(1+t)) = d_1\theta(1) \ .$$

Therefore, compared to classical GIT, in non-algebraic complex geometry, one has to consider more general (non-algebraic!) one-parameter subgroups, which do not exponentiate to morphisms $\mathbb{C}^* \to G$.

To any $\xi \in H(G)$ one can associate a parabolic subgroup $G(\xi)$ of G by

$$G(\xi) := \{g \in G| \lim_{t\to\infty} \exp(t\xi)g\exp(-t\xi) \text{ exists in } G\}$$

The subgroup $G(\xi)$ fits in a short exact sequence

$$\{1\} \longrightarrow U(\xi) \longrightarrow G(\xi) \longrightarrow Z(\xi) \longrightarrow \{1\} \ , \tag{2}$$

where

$$U(\xi) := \{g \in G| \lim_{t\to\infty} \exp(t\xi)g\exp(-t\xi) = e\}$$

is the unipotent radical of $G(\xi)$ and

$$Z(\xi) := \{g \in G| \ \mathrm{ad}_g(\xi) = \xi\} \ ,$$

is the reductive centralizer of ξ.

$G(\xi)$ is the direct product of its subgroups $U(\xi)$, $Z(\xi)$. Note that that the normal subgroup $U(\xi)$ is canonically associated with the parabolic subgroup $G(\xi)$, whereas the subgroup $Z(\xi) \subset G(\xi)$ depends on the choice of the vector ξ.

The Lie algebras of these groups are

$$\mathfrak{u}(\xi) := \bigoplus_{\lambda<0} \mathrm{Eig}([\xi, \cdot], \lambda) \ , \quad \mathfrak{z}(\xi) = z_{\mathfrak{g}}(\xi) = \ker([\xi, \cdot]) \ , \quad \mathfrak{g}(\xi) = \bigoplus_{\lambda\leq0} \mathrm{Eig}([\xi, \cdot], \lambda) \ .$$

Following [Te3] we introduce in $H(G)$ the following important equivalence relation:

Definition 1.1 *We say that two vectors ξ, $\zeta \in H(G)$ are equivalent ($\xi \sim \zeta$) if one of the following equivalent conditions is satisfied:*

1. *$\zeta \in \mathfrak{g}(\xi)$ and there exists $g \in G(\xi)$ such that $\mathrm{ad}_g(\xi) = \zeta$.*

2. *$\zeta - \xi \in \mathfrak{u}(\xi)$.*

3. *$\xi \in \mathfrak{g}(\zeta)$ and there exists $g \in G(\zeta)$ such that $\mathrm{ad}_g(\zeta) = \xi$.*

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4. $\xi - \zeta \in \mathfrak{u}(\zeta)$.

Proposition 1.2 *Let $K \subset G$ be a maximal compact subgroup of G. Then the composition*

$$i\mathfrak{k} \hookrightarrow H(G) \longrightarrow H(G)\big/_{\sim}$$

is a homeomorphism.

In particular, $i\mathfrak{k}$ is a complete system of representatives for the equivalence relation $\sim$.

Corollary 1.3 *If G is a closed reductive subgroup of the reductive group $\hat{G}$, then the inclusion $H(G) \to H(\hat{G})$ induces an injection*

$$H(G)\big/_{\sim_G} \to H(\hat{G})\big/_{\sim_{\hat{G}}}$$

which is a homeomorphism on its image.

Example: Consider the case $G = GL(r, V)$, where V is a finite dimensional complex vector space. Then $H(G) \subset gl(V)$ is the subset of endomorphisms which are diagonalizable and have real eigenvalues. Every such endomorphism ξ defines a filtration

$$V(\xi) = (V_\lambda)_{\lambda \in \operatorname{Spec}(\xi)} \ , \ V_\lambda := \bigoplus_{\nu \leq \lambda} \operatorname{Eig}(\xi, \nu)$$

of V.

The parabolic subgroup $G(\xi)$ consists of those elements $g \in GL(V)$ which leave the filtration $V(\xi)$ invariant. Two elements $\xi, \zeta \in H(G)$ are equivalent if and only if they have the same eigenvalues and define the same filtration.

Let $\alpha : G \times F \to F$ be a holomorphic action of a complex reductive group on a complex manifold F. A *Hamiltonian triple* for α is a triple $\tau = (K, g, \mu)$ consisting of a maximal compact subgroup K of G, a K-invariant Kähler metric g on F, and a moment map μ for the induced K-action on (F, ω_g).

The data of a Hamiltonian triple for the holomorphic action α allows one to construct a *Kählerian quotient* of F by G (see [HH]). As in the algebraic geometric GIT (where the quotient depends on the choice of a linearization of the action), a Kähler quotient depends in general essentially on the choice of the Hamiltonian triple τ.

Definition 1.4 *Let* $\tau = (K, g, \mu)$ *be Hamiltonian triple for the action* α. *A point* $x \in F$ *will be called:*

1. *(symplectically)* τ*-semistable if* $\overline{Gx} \cap \mu^{-1}(0) \neq \emptyset$.

2. *(symplectically)* τ*-stable if* $\mathfrak{g}_x = \{0\}$ *and* $Gx \cap \mu^{-1}(0) \neq \emptyset$.

3. *(symplectically)* τ*-polystable if* $Gx \cap \mu^{-1}(0) \neq \emptyset$.

We will denote by F_τ^{sst}, F_τ^{st}, F_τ^{pst} the subsets of symplectically τ-(semi, poly) stable points in F.

By the results of [HH] one has

Theorem 1.5   *1.* F_τ^{sst} *is open in* F.

2. *There exists a good quotient* $\pi : F_\tau^{\mathrm{sst}} \rightarrow Q_\tau$, *where* Q_τ *is a Hausdorff complex space.*

3. *The closure of every semistable orbit contains a unique polystable orbit.*

4. *The inclusions* $\mu^{-1}(0) \hookrightarrow F_\tau^{\mathrm{pst}} \hookrightarrow F_\tau^{\mathrm{sst}}$ *induce homeomorphisms*

$$\mu^{-1}(0)\big/_K \xrightarrow{\sim} F_\tau^{\mathrm{pst}}\big/_G \xrightarrow{\sim} Q_\tau$$

In particular two semistable orbits have the same image in Q_τ *if and only if the intersection of their closures contains a polystable orbit.*

Therefore the Kähler quotient Q_τ can identified with the symplectic quotient $\mu^{-1}(0)\big/_K$, so this results states that, in the Kähler case, this symplectic quotient (which in general can be very singular) inherits a natural complex space structure.

Let again $\tau = (K, g, \mu)$ be a Hamiltonian triple for α.

Remark 1.6 *For any* $s \in i\mathfrak{k}$, *the map* $t \mapsto \langle \mu(e^{ts}x), -is \rangle$ *is increasing.*

Indeed the derivative of this map at t is just $\|s_{e^{ts}}^{\#}\|^2$.

We put
$$\lambda^s(x) := \lim_{t\to\infty} \langle \mu(e^{ts}x), -is \rangle \ .$$

Note that
$$\lambda^s(x) = \langle \mu(e^{ts}x), -is \rangle + E_g(c_x^s) \tag{3}$$

where $c_x^s : [0, \infty) \rightarrow F$ is the parameterized curve $t \mapsto e^{ts}x$ and E_g stands for the energy with respect to the metric g.

Using the standard equivariance property of the moment map one gets the K-equivariance property of the map λ

$$\lambda^s(x) = \lambda^{\mathrm{ad}_k(s)}(kx) \ \forall s \in i\mathfrak{k} \ , \ k \in K \tag{4}$$

The following stability criterion is well-known (see for instance [Mu1], [Te3]):

Proposition 1.7 *Let $x \in F$. The following conditions are equivalent*

1. *x is symplectically τ-stable.*

2. *$\lambda^s(x) > 0$ for all $s \in i\mathfrak{k} \setminus \{0\}$.*

A point x satisfying the second condition is usually called *analytically τ-stable*. This criterion can be considered as the Kählerian analogue of the Hilbert criterion in GIT. However, this criterion is not satisfactory for our purposes for the following reasons:

1. The map λ is not a purely complex geometric object; it depends on the choice of a Hamiltonian triple for the action α, which is a differential geometric object.

2. As it stands, the analytic stability condition is not a G-invariant condition, because λ is only a K-equivariant function, not a G-equivariant one.

3. The criterion does not work for <u>semi</u>stable points: for general holomorphic actions α on non-compact manifolds the condition $\lambda^s(x) \geq 0$ for all $s \in i\mathfrak{k}$ does <u>not</u> imply that x is symplectically semistable.

The simplest way to overcome these difficulties is to restrict ourselves to *energy complete* Hamiltonian triples (see [Te3]):

Definition 1.8 *A Hamiltonian triple $\tau = (K, g, \mu)$ is called energy complete if for every $(s, x) \in i\mathfrak{k} \times F$ the following implication holds*

$$E_g(c_x^s) < \infty \Rightarrow \lim_{t \to \infty} c_x^s(t) \ \text{exists in } F \ .$$

We refer to [Te3] for the following

Proposition 1.9 *The map $\lambda : i\mathfrak{k} \times F \to \mathbb{R} \cup \{\infty\}$ associated with an energy complete Hamiltonian triple $\tau = (K, g, \mu)$ has a unique extension*

$$\lambda : H(G) \times F \to \mathbb{R} \cup \{\infty\}$$

which is G-equivariant, i. e.

$$\lambda^s(x) = \lambda^{\mathrm{ad}_g(s)}(gx) \ \forall s \in H(G) \ \forall x \in F \forall g \in G \ .$$

This function has the following properties

1. *homogeneity:* $\lambda^{ts}(x) = t\lambda^s(x)$ *for any* $t \in \mathbb{R}_{\geq 0}$, $s \in H(G)$.

2. *parabolic invariance:* $\lambda^s(x) = \lambda^s(hx)$ *for every* $x \in F$, $h \in G(s)$.

3. $\sim$ *invariance:* $\lambda^s(x) = \lambda^\sigma(x)$ *if* $s \sim \sigma$.

4. *semicontinuity: if* $(x_n, s_n)_n \to (x, s)$ *then* $\lambda^s(x) \leq \liminf_{n \to \infty} \lambda^{s_n}(x_n)$.

The G-equivariant extension $\lambda : H(G) \times F \to \mathbb{R} \cup \{\infty\}$ will be called (for historical reasons, see section 1.3.2) the <u>maximal</u> <u>weight</u> <u>function</u> associated with the (energy complete) Hamiltonian triple α.

For our gauge theoretical results we will need the following result

Proposition 1.10 *Let G be a reductive normal subgroup of a reductive group $\hat{G}$, and let $\hat{\alpha} : \hat{G} \times F \to F$ be a holomorphic action.*

Let $K \subset \hat{K}$ be maximal compact subgroups of G and $\hat{G}$ and let $\tau = (K, g, \mu)$ be an energy complete Hamiltonian triple for $\hat{\alpha}|_{G \times F}$ such that μ is $\hat{K}$-equivariant.

Then the corresponding maximal weight function $\lambda : H(G) \times F \to \mathbb{R} \cup \{\infty\}$ is $\hat{G}$-<u>equivariant</u>, i. e. it satisfies one of the following equivalent conditions:

1. $\hat{G}$-*equivariance:* $\lambda^s(x) = \lambda^{\mathrm{ad}_g(s)}(gx)$ *for all* $s \in H(G)$, $g \in \hat{G}$.

2. $\hat{G}$-*parabolic invariance:* $\lambda^s(x) = \lambda^s(hx)$ *for every* $x \in F$, $h \in \hat{G}(s)$.

We denote by G_x the stabilizer of $x \in F$, and by $\mathfrak{g}_x$ its Lie algebra (the *infinitesimal stabilizer* of x). A complex Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ will be called *reductive* if it is the complexification of the Lie algebra of a compact subgroup of G.

Definition 1.11 *Let $\alpha : G \times F \to F$ be a holomorphic a action of a complex reductive group, and let $\lambda : H(G) \times F \to \mathbb{R} \cup \{\infty\}$ be the maximal weight function associated with an energy complete Hamiltonian triple τ. A point $x \in F$ will be called*

1. *analytically τ-semistable, if* $\lambda^s(x) \geq 0$ *for every* $s \in H(G)$.

2. *analytically τ-stable if it is λ-semistable and strict inequality* $\lambda^s(x) > 0$ *holds for all* $s \in H(G) \setminus \{0\}$.

3. *analytically τ-polystable if its infinitesimal stabilizer $\mathfrak{g}_x$ is reductive, it is λ-semistable and $\lambda^s(x) > 0$ for every $s \in H(G)$ which is not equivalent to an element in $\mathfrak{g}_x$.*

Remark 1.12 *If x is analytically τ-semistable then*

$$\lambda^s(x) = \langle \mu(x), -is \rangle = 0 \ \forall s \in i\mathfrak{k}_x \ .$$

Indeed, for $s \in i\mathfrak{k}_x$, the path c_x^s is constant, hence $\lambda^s(x) = \langle \mu(x), -is \rangle$. The right hand term defines a linear form on $i\mathfrak{k}_x$ which takes nonnegative values, hence it vanishes.

Remark 1.13 *Under the assumptions of Definition 1.11, one has:*

1. *x is analytically τ-semistable iff $\lambda^s(x) \geq 0$ for all $s \in i\mathfrak{k}$.*

2. *if $\mathfrak{g}_x = \mathfrak{k}_x^{\mathbb{C}}$, then x is analytically τ-polystable iff $\lambda^s(x) = 0$ for all $s \in i\mathfrak{k}_x$ and $\lambda^s(x) > 0$ for all $s \in i\mathfrak{k} \setminus i\mathfrak{k}_x$.*

Note that if $\mathfrak{g}_x \neq \mathfrak{k}_x^{\mathbb{C}}$, one has no numerical characterization of the polystability of x in terms of elements $s \in i\mathfrak{k}$; on the other hand one can always choose a Hamiltonian triple $\tau' = (K', g', \mu')$ *conjugate* to τ – i. e. a triple of the form

$$(K', g', \mu') = (\mathrm{Ad}_\gamma(K), (\gamma^{-1})^* g, \mathrm{ad}_{\gamma^{-1}}^t \circ \mu \circ \gamma^{-1})$$

such that $\mathfrak{g}_x = \mathfrak{k}_x'^{\mathbb{C}}$; two conjugate triples define the same maximal weight function on $H(G) \times F$ (see [Te3]).

The comparison results in [Te3] show that

Proposition 1.14 *Let $\tau = (K, g, \mu)$ be an energy complete Hamiltonian triple. A point $x \in F$ is symplectically τ-(semi, poly)stable if and only if it is analytically (semi, poly)stable.*

This result should be regarded as a (non-algebraic) *complex geometric Hilbert criterion*. Note that the proof in the <u>semi</u>stable case is rather difficult.

In the next section we will give a short proof for the polystable case, which is based on the continuity method and follows the same strategy as the proof of our universal Kobayashi-Hitchin correspondence.

1.2 The Continuity Method in the finite dimensional case

In this section we give an easy proof based on the continuity method for the following result:

Proposition 1.15 *Let $\tau = (K, g, \mu)$ be an energy complete Hamiltonian triple and let $x \in F$. Then the following conditions are equivalent:*

1. *x is symplectically τ-polystable.*

2. *x is analytically τ-polystable*

We begin with the following simple

Lemma 1.16 *Let μ be a moment map for an action $\alpha : K \times F \to F$ of a compact Lie group on a symplectic manifold F. Then*

$$\mu(x) \in z_{\mathfrak{k}}(\mathfrak{k}_x) \ .$$

Proof: Regarding μ as a map $F \to \mathfrak{k}$ via an ad_k-invariant inner product, we get for any $u \in \mathfrak{k}$

$$d\mu(u_x^\#) = \frac{d}{dt}|_{t=0} \, \mu(e^{tu}(x)) = \frac{d}{dt}|_{t=0} \, \mathrm{ad}_{e^{tu}}(\mu(x)) = [u, \mu(x)] \ . \tag{5}$$

The left hand term vanishes when $u \in \mathfrak{k}_x$. ∎

Proof (of Proposition 1.15):

We start with the simple implication 1. $\Rightarrow$ 2. If x is symplectically τ-polystable, let $x_0 \in Gx \cap \mu^{-1}(0)$. It is easy to see that $\mathfrak{g}_{x_0} = \mathfrak{k}_{x_0}^{\mathbb{C}}$, hence $\mathfrak{g}_{x_0}$ (so also $\mathfrak{g}_x$) are reductive subalgebras. Moreover, one checks easily that $\lambda^s(x_0) = 0$ for $s \in i\mathfrak{k}_{x_0}$ and $\lambda^s(x_0) > 0$ for $s \in i\mathfrak{k} \setminus i\mathfrak{k}_{x_0}$. Therefore x_0 (so also x) is analytically τ-polystable (see [Te3] for details).

Suppose now that x is analytically polystable.

<u>Step 1</u>. Reducing the problem to the case

$$\mathfrak{g}_x \subset z(\mathfrak{g}, G) \ , \tag{6}$$

where
$$z(\mathfrak{g}, G) := \{u \in \mathfrak{g}| \ \mathrm{ad}_g(u) = u \ \forall g \in G\} \ .$$

Note that $z(\mathfrak{g}, G) = z(\mathfrak{g})$ when G is connected.

Put
$$G' := Z_G(\mathfrak{g}_x) = \{g \in G| \ \mathrm{ad}_g(u) = u \ \forall u \in \mathfrak{g}_x\} \ .$$

Since $\mathfrak{g}_x$ is a reductive subalgebra, it is easy to see that G' is a reductive subgroup of G with Lie algebra $\mathfrak{g}' = z_{\mathfrak{g}}(\mathfrak{g}_x)$.

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Consider the restricted action $\alpha' := \alpha|_{G' \times F}$ of G' on F.

The infinitesimal stabilizer $\mathfrak{g}'_x$ of x with respect to α' is $\mathfrak{g}_x \cap \mathfrak{g}'$. Since this subalgebra of $\mathfrak{g}'$ is contained in $\mathfrak{g}_x$, its element are invariant under ad_g for all $g \in G'$, by the definition of this group. Therefore $\mathfrak{g}'_x \subset z(\mathfrak{g}', G')$.

Since $\mathfrak{g}_x$ is a reductive subalgebra, it follows that, replacing τ by a conjugate triple if necessary, we may suppose that $\mathfrak{g}_x = \mathfrak{k}_x^{\mathbb{C}}$.

We have obviously $G' = K'^{\mathbb{C}}$, $\mathfrak{g}' = \mathfrak{k}'^{\mathbb{C}}$, where

$$K' := Z_K(\mathfrak{k}_x) \ , \ \ \mathfrak{k}' = z_{\mathfrak{k}}(\mathfrak{k}_x) \ .$$

The triple $\tau' = (K', g, \mu' := \mu|_{\mathfrak{k}'})$ is an energy complete Hamiltonian triple for α'. It is easy to see that x is analytically τ'-polystable. Suppose we have proved the implication 2. $\Rightarrow$ 1. in the case when (6) holds. It will follow that x is symplectically τ'-polystable, hence there exists $g' \in G'$ such that $\mu'(g'x) = 0$.

We state that, in our case, the relation $\mu'(g'x) = 0$ implies the apparently stronger relation $\mu(g'x) = 0$.

Indeed, note first, that since $g' \in G' = Z_G(\mathfrak{g}_x)$ then $\mathfrak{g}_{g'x} = \mathrm{ad}_{g'}(\mathfrak{g}_x) = \mathfrak{g}_x = \mathfrak{k}_x^{\mathbb{C}}$.

By Lemma 1.16 we get $\mu(g'x) \in z_{\mathfrak{k}}(\mathfrak{k}_x) = \mathfrak{k}'$. But the relation $\mu'(g'x) = 0$ means that the projection of $\mu(g'x)$ on $\mathfrak{k}'$ vanishes. Therefore $\mu(g'x) = 0$.

<u>Step 2</u>. Proving the implication 2. $\Rightarrow$ 1. in the case (6) using the continuity method.

Since $\mathfrak{g}_x \subset z(\mathfrak{g}, G)$, we have

$$\mathfrak{k}_{gx} = \mathfrak{g}_{gx} \cap \mathfrak{k} = \mathrm{ad}_g(\mathfrak{g}_x) \cap \mathfrak{k} = \mathfrak{g}_x \cap \mathfrak{k} = \mathfrak{k}_x \ \forall g \in G \ . \tag{7}$$

We seek an element $g \in G$ such that

$$\mu(gx) = 0 \tag{8}$$

In order to solve this equation, we use the continuity method: we choose a suitable point $x_0 \in Gx$, we solve for $\varepsilon \in (0, 1]$ the perturbed equation

$$i\mu(e^s x_0) + \varepsilon s = 0 \ , \ \ s \in i\mathfrak{k}_{x_0}^{\perp} \tag{9}$$

and then we make $\varepsilon \to 0$.

a) We choose an element $x_0 \in Gx$ such that the perturbed equation (9) has a solution in $i\mathfrak{k}_{x_0}^{\perp}$ for $\varepsilon = 1$.

Set $s_1 := -i\mu(x)$. Put $x_0 := e^{-s_1}x$. Then

$$i\mu(e^{s_1}x_0) + s_1 = i\mu(x) + s_1 = 0 \ .$$

By Remark 1.12 we have $s_1 \in i\mathfrak{k}_x^\perp$, so by (7), we get as required $s_1 \in i\mathfrak{k}_{x_0}^\perp$.

b) We prove that the set

$$T := \{\varepsilon \in (0,1] |\ \exists s \in i\mathfrak{k}_{x_0}^\perp \text{ such that } i\mu(e^s x_0) + \varepsilon s = 0\}$$

is open in $(0,1]$.

We apply the implicit function theorem to the function

$$l : i\mathfrak{k}_{x_0}^\perp \times (0,1] \longrightarrow i\mathfrak{k}_{x_0}^\perp \ , \ \ l(\varepsilon, s) = i\mu(e^s x_0) + \varepsilon s \ .$$

The fact the l takes values in $i\mathfrak{k}_{x_0}^\perp$ follows again from Remark 1.12. Let $s,\ \dot{s} \in i\mathfrak{k}_{x_0}^\perp$ and decompose

$$\dot{s} = \lambda + [k, s] \ , \ \text{with } \lambda \in iz_{\mathfrak{k}}(s) \ , \ k \in z_{\mathfrak{k}}(s)^\perp$$

as in the proof of Proposition 6.10 in the Appendix. Put

$$\sigma := ((d_s \exp)(\dot{s}))\, e^{-s} \ , \ \ \sigma_h := p_{i\mathfrak{k}}(\sigma) \ , \ \ \sigma_a := p_{\mathfrak{k}}(\sigma) \ .$$

Using the known formula (5), we get

$$\frac{\partial l}{\partial s}(\varepsilon, s)(\dot{s}) = id\mu(\sigma_{e^s x_0}^{\#}) + \varepsilon \dot{s} = id\mu((\sigma_h^{\#})_{e^s x_0}) + id\mu((\sigma_a^{\#})_{e^s x_0}) + \varepsilon \dot{s}$$

$$= id\mu((\sigma_h^{\#})_{e^s x_0}) + [\sigma_a, i\mu(e^s x_0)] + \varepsilon \dot{s}$$

At a pair (ε, s) where $l(\varepsilon, s) = 0$, this reduces to

$$\frac{\partial l}{\partial s}(\varepsilon, s)(\dot{s}) = id\mu((\sigma_h^{\#})_{e^s x_0}) - \varepsilon[\sigma_a, s] + \varepsilon \dot{s} \ .$$

Therefore

$$\left\langle \frac{\partial l}{\partial s}(\varepsilon, s)(\dot{s}), \sigma_h \right\rangle = \langle d\mu((\sigma_h^{\#})_{e^s x_0}), -i\sigma_h \rangle - \varepsilon\langle[\sigma_a, s], \sigma_h \rangle + \varepsilon\langle \dot{s}, \sigma_h \rangle =$$

$$\omega(-i(\sigma_h^{\#})_{e^s x_0}, (\sigma_h^{\#})_{e^s x_0}) - \varepsilon\langle[\sigma_a, s], \sigma_h \rangle + \varepsilon\langle \dot{s}, \sigma_h \rangle \geq \|(\sigma_h^{\#})_{e^s x_0}\|^2 + \varepsilon\|\dot{s}\|^2$$

by Proposition 6.10 in Appendix. Therefore $\frac{\partial l}{\partial s}(\varepsilon, s)$ is an invertible operator.

c) We prove that the set T is closed in $(0,1]$.

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Let $(\varepsilon_n)_n$ be a sequence converging to $\varepsilon_0 \in (0,1]$. Taking the inner product of the equation $l(\varepsilon_n, s_{\varepsilon_n}) = 0$ with $s_n := s_{\varepsilon_n}$ we get

$$\langle \mu(e^{s_n} x_0), -is_n \rangle + \varepsilon_n \|s_n\|^2 = 0$$

We know by Remark 1.6 that the function $t \mapsto \langle \mu(e^{ts} x_0), -is \rangle$ is always increasing, so

$$\|s_n\|^2 = -\frac{1}{\varepsilon_n} \langle \mu(e^{s_n} x_0), -is_n \rangle \leq -\frac{1}{\varepsilon_n} \langle \mu(x_0), -is_n \rangle \leq \frac{1}{\varepsilon_n} \|\mu(x_0)\| \|s_n\| \ ,$$

which shows that the sequence $(s_n)_n$ is bounded. The limit of a convergent subsequence will give a solution of the equation $l(\varepsilon_0, \cdot) = 0$.

d) Let $(\varepsilon_n)_n$ be a sequence in $(0,1]$ with $\varepsilon_n \searrow 0$. We prove that any sequence $(s_n)_n$, $s_n \in i\mathfrak{k}^{\perp}_{x_0}$ with $l(\varepsilon_n, s_n) = 0$ is bounded.

Taking a subsequence if necessary, it suffices to prove that the condition $\|s_n\| \to \infty$ leads to a contradiction. Put $t_n := \|s_n\|$, $\sigma_n := \frac{1}{t_n} s_n$. Taking a subsequence again, we may suppose that σ_n converges to an element $\sigma \in i\mathfrak{k}^{\perp}_{x_0}$ whose norm will be of course 1. By the polystability condition, one has

$$\lambda^{\sigma}(x_0) = \lim_{t \to \infty} \langle \mu(e^{t\sigma} x_0), -i\sigma \rangle > 0 \ ,$$

so for t_0 sufficiently large it holds

$$\langle \mu(e^{t_0 \sigma} x_0), -i\sigma \rangle > 0 \ .$$

Fix such a t_0. For <u>all</u> n sufficiently large we will still have

$$\langle \mu(e^{t_0 \sigma_n} x_0), -i\sigma_n \rangle > 0$$

Assuming $t_n \to \infty$, we can find n with the above property such that also $t_n \geq t_0$. By the monotony property Remark 1.6 we would get

$$0 < \langle \mu(e^{t_n \sigma_n} x_0), -i\sigma_n \rangle = \frac{1}{t_n} \langle \mu(e^{s_n} x_0), -is_n \rangle = -\frac{1}{t_n} \varepsilon_n \|s_n\|^2 \ ,$$

which is obviously a contradiction.

Now it suffices to note that the limit of a convergent subsequence of $(s_n)_n$ will be a solution of the equation $\mu(e^s x_0) = 0$. ∎

1.3 Maximal weight functions for linear and projective actions

The explicit form of the maximal weight function associated with a linear or projective actions is well known (see for instance [Mu1]). We explain below this computation for completeness.

1.3.1 Linear actions

Let $\rho : G \to GL(V)$ be a rational linear representation of a connected reductive group G on a finite dimensional complex vector space V.

Let K be a maximal compact subgroup of G with a fixed invariant inner product on its Lie algebra $\mathfrak{k}$, and let g be a K-invariant Hermitian inner product on V. One has a standard moment map for the K action which is given by

$$\mu_0(v) = \rho^*(-\frac{i}{2} v \otimes v^*) \ ,$$

and any other moment map has the form

$$\mu_\tau = \mu_0 - i\tau \ ,$$

where $\tau \in iz(\mathfrak{k})$.

Let $\xi \in i\mathfrak{k}$ and let $V = \oplus_{j=1}^k V_j$ be the decomposition of V into eigenspaces of ξ. In other words $\rho_*(\xi)|_{V_j} = \xi_j \mathrm{id}_{V_j}$, where ξ_j are the pairwise distinct eigenvalues of ξ.

Put

$$V_\xi^\pm := \bigoplus_{\pm\xi_j>0} V_j \ , \ V_\pm^\xi := \bigoplus_{\pm\xi_j\geq0} V_j \ .$$

Let $v \in V$. Decompose v as $v = \sum_j v_j$ with $v_j \in V_j$.

One gets

$$\lambda_\tau^\xi(v) := \lim_{t\to\infty} \langle \mu_\tau(\rho(e^{t\xi})v), -i\xi \rangle = \begin{cases} +\infty & \text{if } \exists j \in \{1,\dots,k\} \text{ such that} \\ & \qquad \xi_j > 0 \text{ and } v_j \neq 0 \ , \\[2mm] \langle \tau, \xi \rangle & \text{otherwise} \end{cases}$$

Suppose for simplicity that $\ker(\rho_*) = \{0\}$.

In this case we conclude that a vector $v \in V$ is τ-stable if and only if for every $\xi \in i\mathfrak{k} \setminus \{0\}$ with $\langle \tau, \xi \rangle \leq 0$ one has $\mathrm{pr}_{V_\xi^+}(v) \neq 0$.

Therefore, the non-stable locus is

$$NS_\tau = \bigcup_{\langle \tau, \xi \rangle \leq 0, \xi \neq 0} V_-^\xi \qquad (*)$$

Let us describe this set in the case when G is a complex torus. Decompose V as

$$V = \bigoplus_{\chi \in R} V_\chi \ ,$$

where $R \subset \mathrm{Hom}(i\mathfrak{k}, \mathbb{R}) \subset \mathfrak{g}^\vee$ is the finite set of weights of the representation ρ, and $V_\chi := \{ v \in V \mid \rho_*(\gamma)(v) = \chi(\gamma)v \ \forall \gamma \in \mathfrak{g} \}$.

For every subset $A \subset R$ put

$$V_A := \bigoplus_{\chi \in A} V_\chi \ ,$$

Since G is a torus, the space V_-^ξ associated with any $\xi \in i\mathfrak{k}$ has the form V_A for some $A \subset R$. Define

$$C_A := \{ \xi \in i\mathfrak{k} \mid V_-^\xi = V_A \} = \{ \xi \in i\mathfrak{k} \mid \chi(\xi) \geq 0 \ \forall \chi \in A, \ \chi(\xi) > 0 \ \forall \chi \in R \setminus A \} \ .$$

Note that the sets C_A give a partition of $i\mathfrak{k}$ in pairwise disjoint polyhedral convex cones. Using $(*)$ we get

$$NS_\tau = \bigcup_{C_A \cap (H_\tau \setminus \{0\})^{\geq 0} \neq \emptyset} V_A \ ,$$

where $H_\tau^{\geq 0}$ denotes the half-space defined by the inequality $\langle \tau, \xi \rangle \geq 0$.

Corollary 1.17 *Suppose that G is a torus. Then there exists a finite set $\Phi_\tau \subset i\mathfrak{k}$ such that the following conditions become equivalent:*

1. *v is τ-stable,*

2. *$\lambda_\tau^\xi(v) > 0$ for all $\xi \in \Phi_\tau$,*

3. *$v \notin \bigcup_{\xi \in \Phi} V_-^\xi$.*

Proof: The non-empty intersections $C_A \cap (H_\tau \setminus \{0\})$ define a finite partition of the pointed half-space $H_\tau^{\geq 0} \setminus \{0\}$. Take a point in every set of this partition. ∎

1.3.2 Projective actions

Let $\rho : G \to GL(V)$ be a rational linear representation of a complex reductive group G, and let $\alpha : G \times \mathbb{P}(V) \to \mathbb{P}(V)$ be the induced projective action.

The Fubini-Study Kähler form on $\mathbb{P}(V)$ associated with a Hermitian metric h on V is $\frac{i}{2\pi}\partial\bar{\partial}\log \| \cdot \|^2 = \frac{1}{4\pi}dd^c \log \| \cdot \|^2$.

Let $K \subset G$ be a maximal compact subgroup which acts isometrically on the Hermitian space (V, h). The standard moment map for the K-action on $\mathbb{P}(V)$ is (see [Ki])

$$\mu([v]) = \rho^* \left[-\frac{i}{2\pi}\frac{v \otimes \bar{v}}{\|v\|^2} \right]$$

The maximal weight of the Hermitian endomorphism $f \in \mathrm{Herm}(V)$ with respect to a vector $v \in V$ is defined

$$\lambda_m(f, v) := \max \left\{ \lambda \in \mathrm{Spec}(f)|\ \mathrm{pr}_{\mathrm{Eig}(f,\lambda)}(v) \neq 0 \right\} \ .$$

If f is associated with a one parameter subgroup of $GL(V)$ (i. e. it has the form $d_1\varphi(1)$, where $\varphi : \mathbb{C}^* \to GL(V) \in \mathrm{Hom}(\mathbb{C}^*, GL(V))$), then all its eigenvalues are integers, so in this case $\lambda_m(f, \cdot)$ takes integer values. In this case we put $\lambda_m(\varphi, \cdot) := \lambda_m(d_1\varphi(1), \cdot)$. Writing

$$\varphi(\zeta) = \sum_{n \in R(\varphi)} \zeta^n \mathrm{id}_{V_n^\varphi}$$

with $V = \bigoplus_{n \in R(\varphi)} V_n^\varphi$, we see that

$$\lambda_m(\varphi, [v]) = \max \left\{ n \in R(\varphi)|\ \mathrm{pr}_{V_n^\varphi}(v) \neq 0 \right\} \ ,$$

which is just the maximal weight of φ with respect to $[v]$ occurring in the algebraic geometric Hilbert criterion.

Let $\xi \in i\mathfrak{k}$. One has

$$\lambda^\xi([v]) := \lim_{t\to\infty} \langle \mu(\rho(e^{t\xi})v), -i\xi \rangle = \frac{1}{2\pi} \lim_{t\to\infty} \left\langle \left[\frac{e^{t\xi}v \otimes \overline{e^{t\xi}v}}{\|e^{t\xi}v\|^2} \right], \rho_*(\xi) \right\rangle =$$

$$= \frac{1}{2\pi}\lambda_m(\rho_*(\xi), v)$$

Therefore, up to a positive constant, $\lambda^\xi([v])$ is just the maximal weight of the Hermitian endomorphism $\rho_*(\xi)$ with respect to $[v]$.

The point is that, for a rational representation ρ, the analytic (semi)stability of a point $[v] \in \mathbb{P}(V)$ can be checked by verifying the corresponding inequality only for one-parameter subgroups $\mathbb{C}^* \to G$ (see [Ki]). In other words, in this case one has

Analytic Stability = GIT Stability = Symplectic Stability .

2 A "universal" complex geometric classification problem

2.1 Oriented holomorphic pairs

Let
$$1 \longrightarrow G \xrightarrow{\ j\ } \hat{G} \longrightarrow G_0 \longrightarrow 1$$
be an exact sequence of complex reductive Lie groups; the closed normal subgroup G of $\hat{G}$ will be called the symmetry group of the variables and the quotient group G_0 will be called the symmetry group of parameters (see the discussion in the Introduction).

Let X be a connected compact complex manifold, $\hat{Q}$ a differentiable principal $\hat{G}$-bundle over X, and set $Q_0 := \hat{Q} \times_{\hat{G}} G_0$. Let F be a Kähler manifold and $\hat{\alpha}$ a holomorphic action of $\hat{G}$ on F. Consider the associated bundle
$$E := \hat{Q} \times_{\hat{G}} F \ .$$

We fix a holomorphic structure J_0 on Q_0, and we denote by $\mathcal{Q}_0$ the resulting holomorphic bundle. We state the following universal complex geometric classification problem:

Classify the pairs $(\hat{J}, \varphi)$, where $\hat{J}$ is a holomorphic structure on the bundle $\hat{Q}$ which induces J_0 on Q_0 and φ is a $\hat{J}$-holomorphic section in E. The classification is considered up to isomorphy defined by the natural action of the complex gauge group
$$\mathcal{G} := \mathrm{Aut}_{Q_0}(\hat{Q}) = \Gamma(X, \hat{Q} \times_{\mathrm{Ad}_{\hat{G}}} G) \ .$$

A pair $(\hat{J}, \varphi)$ as above will be called *a holomorphic pair of type* $(\hat{Q}, J_0, \hat{\alpha})$.

A possible answer to the classification problem above can be given by restricting our attention to the subspace of *simple* pairs. The concept of simple pair can be defined in our general framework as follows:

For any pair $(\hat{J}, \varphi)$ we define the *space of its infinitesimal automorphisms* (its *infinitesimal stabilizer*) by
$$\mathfrak{g}_{\hat{J}, \varphi} := \{ v \in A^0(\hat{Q} \times_{\mathrm{ad}} \mathfrak{g}) \mid \bar{\partial}_{\hat{j}} v = 0, \ v^{\#} \circ \varphi = 0 \} \ ,$$
where $v^{\#}$ denotes the vertical vector field on E defined by v.

Definition 2.1 *The pair $(\hat{J}, \varphi)$ is called simple if* $\mathfrak{g}_{\hat{J}, \varphi} = \{0\}$.

Using non-linear versions of the methods of [LO], [LT] one should be able to endow the moduli space of simple holomorphic pairs with the structure of a

(possibly non-Hausdorff, possibly singular) complex orbifold (see [Su] for the case of standard holomorphic pairs).

A more involved approach to the classification problem is based on the concept of stability. The data of a Hermitian metric g on X allows us to introduce a concept of stability for holomorphic pairs of a fixed type $(\hat{Q}, J_0, \hat{\alpha})$, and to construct the corresponding moduli space of g-stable pairs. This moduli space will be a Hausdorff open subspace in the moduli orbifold of simple pairs.

Remark: In algebraic geometry and classical complex geometry one avoids using differential geometric objects (as is our fixed differentiable bundle $\hat{Q}$), so it is worth to note that there is a purely complex geometric (algebraic geometric) equivalent formulation of our classification problem.

Fix an algebra morphism $\hat{\mathfrak{c}} : H^*(B\hat{G}, \mathbb{Z}) \to H^*(X, \mathbb{Z})$. Classify the triples $(\hat{\mathcal{Q}}, \iota, \varphi)$, where $\hat{\mathcal{Q}}$ is a holomorphic $\hat{G}$ bundle on X with fixed characteristic morphism $\hat{\mathfrak{c}}$, ι is a holomorphic isomorphism $\iota : \hat{\mathcal{Q}} \times_{\hat{G}} G_0 \to \mathcal{Q}_0$ and φ is a holomorphic section of $\hat{\mathcal{Q}} \times_{\hat{G}} F$. The classification is considered up to isomorphism of holomorphic $\hat{G}$-bundles.

A triple $(\hat{\mathcal{Q}}, \iota, \varphi)$ as above will be called a holomorphic triple of type $(\hat{\mathfrak{c}}, \mathcal{Q}_0, \hat{\alpha})$. The algebraic version can be obtained by replacing everywhere the word "holomorphic" by "algebraic". This terminology was used in [OST].

One can prove easily that a "moduli set" of holomorphic triples of a fixed type is a discrete disjoint union of "moduli sets" of oriented holomorphic pairs.

2.2 The stability condition for universal oriented holomorphic pairs

2.2.1 The degree of a meromorphic L-reduction with respect to an ad-invariant linear form

Let G be a complex reductive group, and let L be a parabolic subgroup of G. When G is connected it follows that L has the form $L = G(s)$, where $s \in H(G)$. In the general case we will consider only parabolic subgroups of the the form $G(s)$ so, for us, a parabolic subgroup will always be associated to an element in $s \in H(G)$. We know that L fits in the exact sequence

$$1 \to U(L) \to L \to Z(L) \to 1$$

which is canonically associated with L, i. e. it does not depend on the presentation $L = G(s)$. $U(L)$ is just the unipotent radical of L, and $Z(L)$ is the canonical reductive quotient of L.

Let $\psi : \mathfrak{l} \to \mathbb{C}$ be an ad_L-invariant linear form. Such a form must be a Lie-algebra morphism[2], i. e. it must fulfill

$$\psi([a,b]) = 0 \ \forall \ a, \ b \in \mathfrak{l} \ .$$

Recall that

$$\mathfrak{u}(L) = \mathfrak{u}(s) := \bigoplus_{\lambda < 0} \mathrm{Eig}([s, \cdot], \lambda) \ , \ \mathfrak{z}(L) = \mathfrak{z}(s) = z_{\mathfrak{g}}(s) \ ,$$

and that $[s, \mathfrak{u}(s)] = \mathfrak{u}(s)$. This shows that $\psi|_{\mathfrak{u}(s)} = 0$. Therefore,

Remark 2.2 *Any ad_L-invariant linear form $\psi : \mathfrak{l} \to \mathbb{C}$ is induced by an $\mathrm{ad}_{Z(L)}$-invariant linear form $\mathfrak{z}(L) \to \mathbb{C}$ on its canonical reductive quotient.*

Let now Z by any complex reductive group. Let C be a maximal compact subgroup of Z and $T \subset C$ a maximal torus. Consider the root decomposition

$$\mathfrak{z} = \mathfrak{t}^{\mathbb{C}} \oplus (\bigoplus_{r \in R \setminus \{0\}} \mathfrak{z}_r)$$

of the Lie algebra $\mathfrak{z}$. For any $r \in R \setminus \{0\}$, one has obviously $\mathfrak{z}_r \subset [\mathfrak{t}^{\mathbb{C}}, \mathfrak{z}_r]$. Therefore,

Remark 2.3 *If $\psi : \mathfrak{z} \to \mathbb{C}$ is an ad_Z-invariant form, then $\psi|_{\mathfrak{z}_r} = 0$ for every $r \in R \setminus \{0\}$.*

Definition 2.4 *An ad-invariant form $\psi : \mathfrak{z} \to \mathbb{C}$ will be called real if for any (and hence for every) maximal compact subgroup C of Z, the restriction $\psi|_{i\mathfrak{c}}$ takes real values.*

Definition 2.5 *A non-degenerate ad_G-invariant symmetric bilinear form*

$$h : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$$

will be called invariant <u>complex</u> <u>inner</u> <u>product</u> of <u>Euclidean</u> <u>type</u> if it induces an Euclidean inner product on any (and hence on all) subspaces of the form $i\mathfrak{k}$, where $\mathfrak{k}$ is the Lie algebra of a maximal compact subgroup K.

In the semisimple case, the Killing form is such an invariant complex inner product of Euclidean type. We can always find such an complex inner product by choosing an ad_K-invariant real inner product on any subspace $i\mathfrak{k}$ and

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[2]For a connected Lie group, the converse is also true: a $\mathbb{C}$-valued Lie algebra morphism is an ad-invariant form.

extending it in a $\mathbb{C}$-bilinear way. However, this notion should be regarded as a complex geometric one (which does not require a distinguished maximal compact subgroup).

Example: Let h be an invariant complex inner product of Euclidean type on $\mathfrak{g}$. Any element $s \in H(G)$ defines a real $\mathrm{ad}_{G(s)}$-invariant form $h(s) : \mathfrak{g}(s) \to \mathbb{C}$ given by $u \mapsto h(s, u)$.

Note that the restriction of h to $\mathfrak{g}(s)$ is <u>not</u> non-degenerate. Its kernel is $\mathfrak{u}(s)$. Indeed, for any u, $v \in \mathfrak{g}(s)$, one has

$$h(u, v) = h(\mathrm{ad}_{\exp(ts)}(u), \mathrm{ad}_{\exp(ts)}(v)) = h(\mathrm{pr}_{\mathfrak{z}(s)}(u), \mathrm{pr}_{\mathfrak{z}(s)}(v)) \ ,$$

because $\mathrm{ad}_{\exp(ts)}(u) \to \mathrm{pr}_{\mathfrak{z}(s)}(u)$ as $t \to \infty$. Therefore, the formula for $h(s)$ reads

$$h(s)(u) = h(s, \mathrm{pr}_{\mathfrak{z}(s)}(u)) \ ,$$

which is obviously $\mathrm{ad}_{G(s)}$-invariant, because $G(s) \to Z(s)$ is a group morphism, h is ad-invariant, and s is $\mathrm{ad}_{Z(s)}$-invariant.

By the Chern Weil theory, any ad_Z-invariant linear form ψ on its Lie algebra $\mathfrak{z}$ defines a characteristic class $c_\psi(W)$ in the 2 -cohomology of the base of any Z-bundle W. More precisely, if A is a a connection on W, one sets

$$c_\psi(W) = \left[\psi(\frac{i}{2\pi} F_A) \right]_{\mathrm{DR}} \ .$$

For a L-bundle Λ we put

$$c_\psi(\Lambda) := c_\psi(\Lambda \times_L Z(L)) \ .$$

Definition 2.6 *The degree of a Z-bundle W (respectively of a L-bundle Λ) on a compact n-dimensional Kähler manifold (X, g) with respect to the real ad-invariant form $\psi : \mathfrak{z} \to \mathbb{C}$ (respectively $\psi : \mathfrak{z}(L) \to \mathbb{C}$) is defined by*

$$\deg_g(W, \psi) := \langle c_\psi(W) \cup [\omega_g]^{n-1}, [X] \rangle \ (\deg_g(\Lambda, \psi) := \deg_g(\Lambda \times_L Z(L)))$$

In the non-Kähler case, one can still define the degree with respect to ψ of a holomorphic Z bundle (L-bundle), but the obtained invariant will no longer be a topological invariant. One has to use the same strategy as in the well known case of line bundles on Gauduchon manifolds.

Let g be a Gauduchon metric on X and let $\mathcal{W}$ be a *holomorphic Z-bundle* on X. Choose a maximal compact subgroup C of Z, an C-reduction R of $\mathcal{W}$ and let A_R its Chern connection (see section 6.1). The point is that, as in the case of line bundles, the following important result holds:

Lemma 2.7 *The real closed 2-form $\psi(\frac{i}{2\pi}F_{A_R})$ is independent of the chosen C-reduction R up to a $i\partial\bar{\partial}$-exact form. Therefore, the $i\partial\bar{\partial}$-cohomology class defined by $\psi(\frac{i}{2\pi}F_{A_R})$ depends only on the holomorphic Z-bundle $\mathcal{W}$.*

Proof: Let R_0 and R be two C-reductions of $\mathcal{W}$. We can write

$$R = e^{-\frac{s}{2}}(R_0) \ ,$$

where $s \in A^0(R_0 \times_{\mathrm{ad}} i\mathfrak{c})$. By Corollary 6.3, one has

$$F_{A_R} = F_{A_{R_0}} + \bar{\partial}(e^{-s}\partial_{A_{R_0}}(e^s)) \ ,$$

It is easy to see that

$$\psi(e^{-s}\partial_{A_{R_0}}(e^s)) = \partial(\psi(s)) \ .$$

This follows easily using Proposition 6.12 and formula (86), taking into account Remark 2.3. Therefore

$$\psi(F_{A_R}) = \psi(F_{A_{R_0}}) + \bar{\partial}\partial\psi(s) \ .$$

■

This allows us to define

Definition 2.8 *Let $\mathcal{W}$ be a holomorphic Z-bundle, $\mathcal{L}$ a holomorphic L-bundle on a compact n-dimensional Gauduchon manifold (X, g), and let $\psi : \mathfrak{z} \to \mathbb{C}$ (respectively $\psi : \mathfrak{z}(L) \to \mathbb{C}$) be a real* ad-*invariant form. Let C be a maximal compact subgroup of Z (respectively $Z(L)$). We define*

$$c_\psi(\mathcal{W}) = \left[\psi(\frac{i}{2\pi}F_{A_R})\right] \in {}^{Z^{1,1}(X,\mathbb{R})}\big/_{i\partial\bar{\partial}A^0(X,\mathbb{R})} \ , \ c_\psi(\mathcal{L}) := c_\psi(\mathcal{L} \times_L Z(L))$$

$$\deg(\mathcal{W}, \psi) := \langle c_\psi(\mathcal{W}), \omega_g^{n-1}\rangle = \int_X \psi(\frac{i}{2\pi}F_{A_R}) \wedge \omega_g^{n-1} \ ,$$

$$\deg(\mathcal{L}, \psi) := \deg(\mathcal{L} \times_L Z(L), \psi) \ ,$$

where A_R is the Chern connection of any C-reduction of $\mathcal{W}$ (of $\mathcal{L} \otimes_L Z(L)$).

We will need an extension of the degree map to meromorphic L-reductions of a holomorphic G-bundle. We begin with the following

Definition 2.9 *Let L be a parabolic subgroup of a complex reductive group G and $\mathcal{Q}$ a holomorphic G-principal bundle over a complex manifold X. A meromorphic reduction of G to L is a meromorphic section in the associated bundle*

$$\mathcal{Q}_L := \mathcal{Q}/_L = \mathcal{Q} \times_G \left(G/_L \right) ,$$

i. e. a closed reduced irreducible subspace $\rho \subset \mathcal{Q}_L$ which generically projects isomorphically onto X via the bundle projection $q_L : \mathcal{Q}_L \to X$.

Since the fibre G/L of $\mathcal{Q}_L$ is a projective manifold, it follows easily that the maximal open subset $X_\rho \subset X$ over which ρ is the graph of a holomorphic section is the complement of a Zariski closed subset of codimension at least 2. On X_ρ ρ defines a holomorphic L-reduction $\mathcal{Q}^\rho \subset \mathcal{Q}|_{X_\rho}$.

Example: Let $\mathcal{Q}$ be a holomorphic $GL(r, \mathbb{C})$-bundle and $\mathcal{E}$ the associated holomorphic vector bundle. The data of a meromorphic reduction of $\mathcal{Q}$ to the parabolic group

$$L := \left\{ \left. \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \right| A \in GL(p, \mathbb{C}),\ B \in M(p, r - p),\ C \in GL(r - p, \mathbb{C}) \right\}$$

is equivalent to the data of a rank p subsheaf $\mathcal{F} \subset \mathcal{E}$ with torsion free quotient $\mathcal{H}$. Such a subsheaf is necessarily reflexive and defines a subbundle on the subset $X \setminus \mathrm{Sing}(\mathcal{H})$. Given such a subsheaf $\mathcal{F}$, the corresponding meromorphic section ρ is given by

$$\rho := \overline{\mathrm{im}(f)}$$

where $f : X \setminus \mathrm{Sing}(\mathcal{H}) \to \mathbb{G}r_p(\mathcal{E})$ is defined by the holomorphic subbundle $\mathcal{F}|_{X \setminus \mathrm{Sing}(\mathcal{H})}$ of $\mathcal{E}|_{X \setminus \mathrm{Sing}(\mathcal{H})}$.

Suppose now that $\mathcal{Q}$ is a holomorphic G-bundle on the compact Gauduchon manifold (X, g), L a parabolic subgroup of G, $\rho \subset \mathcal{Q}_L$ a meromorphic L-reduction of $\mathcal{Q}$ and $\psi : \mathfrak{z}(L) \to \mathbb{C}$ a real $\mathrm{ad}_{Z(L)}$-invariant form. We denote by $q_L : \mathcal{Q}_L \to X$ the natural projection and by $\iota_\rho : \rho \hookrightarrow \mathcal{Q}_L$ the inclusion map.

We claim that the assignment

$$A^{1,1}(\mathcal{Q}_L) \ni \varphi \mapsto \int_{\rho \setminus \mathrm{Sing}(\rho)} \varphi \wedge q_L^*(\omega_g^{n-1}) \tag{10}$$

defines a $\partial\bar{\partial}$-closed current of bidimension $(1, 1)$ on $\mathcal{Q}_L$. Indeed, let $\mathfrak{R} \xrightarrow{\alpha} \rho$ any desingularization of ρ. Then the assignment

$$A^{1,1}(\mathcal{Q}_L) \ni \varphi \mapsto \int_{\mathfrak{R}} (\iota_\rho \circ \alpha)^*(\varphi) \wedge (q_L \circ \alpha)^*(\omega_g^{n-1}) \tag{11}$$

obviously defines a $\partial\bar{\partial}$-closed current of bidimension $(1,1)$ on $\mathcal{Q}_L$, because the pull-back $(q_L \circ \alpha)^*(\omega_g^{n-1})$ is a $\partial\bar{\partial}$-closed form on the smooth manifold $\mathfrak{R}$.

But the assignment (11) coincides with (10) – which is independent of the desingularization α – because $\rho \setminus \mathrm{Sing}(\rho)$ is identified via α with an open subset of total measure of $\mathfrak{R}$.

We will denote the current defined by the formulae (10), (11) by $q_L^*(\omega_g^{n-1})|_\rho$.

Since, by definition, $\mathcal{Q}_L$ is the base of the holomorphic L-bundle $\mathcal{Q} \to \mathcal{Q}_L$, we can define

Definition 2.10

$$\deg(\rho, \psi) := \left\langle c_\psi(\mathcal{Q} \to \mathcal{Q}_L), q_L^*(\omega_g^{n-1})|_\rho \right\rangle \ .$$

In the presence of a K-reduction $P \subset \mathcal{Q}$ of the G-bundle $\mathcal{Q}$, one can give an explicit formula for the degree map.

Note first that the intersection $K_L := L \cap K$ projects isomorphically onto a maximal compact subgroup of $Z(L)$ via the canonical epimorphism $L \to Z(L)$. This follows from formula (94) in Appendix.

Therefore, $P \to \mathcal{Q}_L \simeq P/K_L$ can be regarded as a reduction of the associated $Z(L)$-bundle $\mathcal{W} := \mathcal{Q} \times_L Z(L) \to \mathcal{Q}_L$ to its maximal compact subgroup K_L.

It is not difficult to express the Chern connection B of the K_L-bundle $P \to \mathcal{Q}_L$ in terms of the Chern connection A of the original K-bundle $P \to X$. Denoting by ω_A, ω_B the corresponding connection forms, one has

$$\omega_B = \mathrm{pr}_{\mathfrak{l} \cap \mathfrak{k}}(\omega_A)$$

where $\mathrm{pr}_{\mathfrak{l} \cap \mathfrak{k}}$ denotes the projection $\mathfrak{k} \to \mathfrak{k}_L$ defined by the canonical ad_{K_L}-invariant decomposition

$$\mathfrak{k} = \mathfrak{k}_L \oplus \mathfrak{k}_L^\perp \ , \tag{12}$$

where $\mathfrak{k}_L^\perp := \mathrm{pr}_{\mathfrak{k}}(\mathfrak{u}(L))$. Therefore, on P one can write

$$\omega_A = \omega_B + \alpha \ , \ \mathrm{pr}_{\mathfrak{k}_L}\Omega_A = \Omega_B + \frac{1}{2}\mathrm{pr}_{\mathfrak{k}_L}[\alpha \wedge \alpha] \ ,$$

where $\alpha \in A^1(P, \mathfrak{k}_L^\perp)$ is the $\mathfrak{k}_L^\perp$-component of ω_A. Regarding the forms in the second formula as bundle valued forms on the base $\mathcal{Q}_L$ one gets

$$\mathrm{pr}_{P \times_{K_L} \mathfrak{k}_L}[q_L^*(F_A)] = F_B + \frac{1}{2}\mathrm{pr}_{P \times_{K_L} \mathfrak{k}_L}[a \wedge a] \ ,$$

where $a \in A^1(\mathcal{Q}_L, P \times_{K_L} \mathfrak{k}_L^\perp)$ is the bundle valued 1-form on $\mathcal{Q}_L$ associated with α.

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Using the definitions above, we get

$$\deg(\rho, \psi) = \int\limits_{\rho \setminus \mathrm{Sing}(\rho)} \psi(\frac{i}{2\pi} F_B) \wedge q_L^*(\omega_g^{n-1}) = \int\limits_{X_\rho} \psi(\frac{i}{2\pi} \rho^*(F_B)) \wedge \omega_g^{n-1} \ ,$$

where ρ^* is the pull-back associated with the holomorphic map $X_\rho \to \mathcal{Q}_L$ defined by ρ. The point is that the form $\rho^*(F_B)$ is just the curvature of the connection A_ρ induced by A on the K_L-subbundle $Q^\rho \cap P \subset P|_{X_\rho}$, whereas $a_\rho := \rho^*(a)$ is just the second fundamental form of this subbundle. Therefore we get the following

Proposition 2.11 *If g be a Gauduchon metric on X, then*

$$\deg_g(\rho, \psi) = \int\limits_X \psi(\frac{i}{2\pi} F_{A_\rho}) \wedge \omega_g^{n-1} \ , \tag{13}$$

where A_ρ is the connection induced by the Chern connection A of any K-reduction $P \subset Q$ on the K_L-subbundle $P^\rho := Q^\rho \cap P \subset P|_{X_\rho}$.

Using the canonical decomposition $\mathfrak{g} = \mathfrak{z}(L) \oplus \mathfrak{z}(L)^\perp$ obtained by complexifying the decomposition (12), one can extend ψ in a natural way to a linear form $\psi : \mathfrak{g} \to \mathbb{C}$ (which of course is not ad_G-invariant in general).

Therefore the degree formula becomes

$$\deg_g(\rho, \psi) = \int\limits_X \left[\psi(\frac{i}{2\pi} F_A) - \psi(\frac{i}{4\pi} [a_\rho \wedge a_\rho]) \right] \wedge \omega_g^{n-1} \ , \tag{14}$$

where a_ρ is the second fundamental form of $Q^\rho \cap P$ in $P|_{X_\rho}$.

The degree map can be extended to the case of an arbitrary Hermitian metric g in the following way (see [LT] for the case of vector bundles):

Recall first that the maximum principle holds for the operator $P = i\Lambda_g \bar{\partial} \partial$ for any Hermitian metric g. Consequently, one gets a direct sum decomposition

$$\mathcal{C}^\infty(X, \mathbb{R}) = \mathbb{R} \oplus P(\mathcal{C}^\infty(X, \mathbb{R}))$$

which is L^2-orthogonal when g is Gauduchon. Dualizing, we get a direct sum decomposition

$$\mathcal{D}'(X, \mathbb{R}) = \mathbb{R} \oplus P(\mathcal{D}'(X, \mathbb{R}))$$

Using this decomposition, we define the operator $I_g : \mathcal{D}'(X, \mathbb{R}) \to \mathbb{R}$ (the generalized integral) by

$$I_g(\varphi) := Vol_g(X)\mathrm{pr}_{\mathbb{R}}(\varphi) \ .$$

I_g vanishes always on $\mathrm{im}(P)$ and coincides with the standard integral when g is Gauduchon.

Remark 2.12 *(The Hermitian case) Let g be a Hermitian metric. Replacing the usual integral by the operator I_g in the formulae (13), (14) one gets a well defined holomorphic invariant $\deg_g(\rho, \psi)$ for pairs (ρ, ψ) consisting of a real ad-invariant linear form ψ on $\mathfrak{z}(\mathfrak{l})$ and a meromorphic reduction ρ of $\mathcal{Q}$ to the parabolic subgroup L.*

The degree with respect to a Hermitian metric g coincides with the degree with respect to the unique Gauduchon metric g' in its conformal class defined by the condition $\int_X (g'/g)^{n-1} vol_g = Vol_g(X)$. Therefore, the properties of the degree map associated with general Hermitian metrics reduce easily to the Gauduchon case.

2.2.2   Stability, Semistability, Polystability

We come back to our gauge theoretical problem, so consider a differentiable principal $\hat{G}$-bundle $\hat{Q}$ on a compact Hermitian manifold (X, g).

Let $\hat{\alpha} : \hat{G} \times F \to F$ be a holomorphic action, G a closed normal subgroup of $\hat{G}$. We choose a Kähler structure g_F on F, a maximal compact subgroup K of G acting isometrically on (F, g_F), a maximal compact subgroup $\hat{K}$ of $\hat{G}$ which contains K, and a $\hat{K}$-equivariant moment map μ for the induced K-action. We will suppose that the Hamiltonian triple $\sigma = (K, g_F, \mu)$ is energy complete, so that it defines a generalized maximal weight function $\lambda : F \times H(G) \to \mathbb{R} \cup \{\infty\}$ with the fundamental properties listed in Proposition 1.9 and 1.10. We also recall that λ depends only on the conjugacy class of σ (see section 1.1).

Let E be the associated bundle $E := \hat{Q} \times_{\hat{G}} F$. Consider again $\zeta \in H(G)$ and a $\hat{G}(\zeta)$-reduction $\hat{Q}^\rho \subset \hat{Q}|_{X_\rho}$ defined on an open subset $X_\rho \subset X$. For points $x \in X_\rho$, $e \in E_x$ we put

$$\lambda^\zeta(e, \rho) := \lambda^\zeta(\varphi_q(e)) \ ,$$

where $\varphi_q : E_x \to F$ is the holomorphic isomorphism defined by an element $q \in \hat{Q}_x^\rho$. By Proposition 1.10 it follows that the term on the right does not depend on the choice of $q \in \hat{Q}_x^\rho$.

We fix an ad-invariant complex inner product h of Euclidean type on $\hat{\mathfrak{g}}$ (see Definition 2.5).

Definition 2.13 *Fix a holomorphic structure J_0 on Q_0. A holomorphic pair $(\hat{J}, \varphi)$ of type $(\hat{Q}, J_0, \hat{\alpha})$ is called σ-<u>semistable</u> if, for every $\xi \in H(G)$ and for every meromorphic reduction ρ of the holomorphic bundle $\hat{\mathcal{Q}}_{\hat{j}}$ to $\hat{G}(\xi)$ it holds*

$$\frac{2\pi}{(n-1)!} \deg_g(\rho, h(\xi)) + \int\limits_{X_\rho} \lambda^\xi(\varphi, \rho) vol_g \geq 0 \ .$$

A pair $(\hat{J}, \varphi)$ of type $(\hat{Q}, J_0, \hat{\alpha})$ is called σ-_stable_ if it is σ-semistable and strict inequality holds for any $\xi \in H(G) \setminus \{0\}$.

The left hand term in the inequality in Definition 2.13 will be called the total maximal weight of the pair $(\hat{J}, \varphi)$ with respect to the pair (ξ, ρ).

Remark 2.14 *In general, the (semi)stability condition depends on the Hermitian metric g, the Hamiltonian triple $\sigma = (K, g_F, \mu)$ and the invariant complex inner product of Euclidean type h.*

Note that, for any $\hat{g} \in \hat{G}$, the translation $R_{\hat{g}} : \hat{Q} \to \hat{Q}$ defines a biholomorphic diffeomorphism $\hat{\mathcal{Q}}/\hat{G}(\psi) \to \hat{\mathcal{Q}}/\hat{G}(\mathrm{ad}_{\hat{g}^{-1}}(\psi))$, hence it assigns to a $\hat{G}(\psi)$ meromorphic reduction ρ of $\hat{\mathcal{Q}}$ a $\hat{G}(\mathrm{ad}_{g^{-1}}(\psi))$ meromorphic reduction $\rho\hat{g}$. The corresponding total maximal weights of a pair $(\hat{J}, \varphi)$ coincide, so we get the following important

Remark 2.15 *It suffices to check the (semi)stability condition for a single representative in every equivalence class of $H(G)$ modulo the adjoint action $\mathrm{ad}_{\hat{G}}$ of the group $\hat{G}$.*

In particular it suffices to check this inequality for any element $\psi \in iC$, where $C \subset \mathfrak{t}$ is a closed Weyl chamber in the Lie algebra $\mathfrak{t}$ of a maximal torus of a maximal compact subgroup $K \subset G$.

We recall that a Lie subalgebra $\mathfrak{b} \subset \mathfrak{g}$ is called reductive if it is the complexification of the Lie algebra of a compact subgroup of G.

Definition 2.16 *Let $\hat{Q}$ be a $\hat{G}$-bundle on a connected compact complex manifold X. A finite dimensional Lie subalgebra $\mathfrak{u} \subset A^0(\hat{Q} \times_{\mathrm{ad}} \mathfrak{g})$ will be called _reductive_ if there exists a reductive Lie subalgebra $\mathfrak{b} \subset \mathfrak{g}$ and a $Z_{\hat{G}}(\mathfrak{b})$-reduction $\hat{Z} \hookrightarrow \hat{Q}$ of $\hat{Q}$ such that $\mathfrak{u}$ is the image of $\mathfrak{b}$ via the obvious morphism*

$$\mathfrak{b} \hookrightarrow A^0(\hat{Z} \times_{\mathrm{ad}} \mathfrak{b}) \to A^0(\hat{Z} \times_{\mathrm{ad}} \mathfrak{g}) = A^0(\hat{Q} \times_{\mathrm{ad}} \mathfrak{g}) .$$

Such a reduction gives an isomorphism $-_{\hat{Z}} : \mathfrak{b} \to \mathfrak{u}$. Note that the reduction $\hat{Z}$ can be recovered from the isomorphism $b_{\hat{Z}}$, because

$$\hat{Z} = \{q \in \hat{Q} | \ b_{\hat{Z}}(q) = b \ \forall b \in \mathfrak{b}\} . \tag{15}$$

In particular one has

Remark 2.17 *If $\hat{\mathcal{Q}}$ is a holomorphic $\hat{G}$-bundle on X and $\mathfrak{u} \subset H^0(X, \hat{\mathcal{Q}} \times_{\mathrm{ad}} \mathfrak{g})$ is a reductive Lie subalgebra consisting of holomorphic sections, then any $Z_{\hat{G}}(\mathfrak{b})$-reduction $\hat{Z} \subset \hat{\mathcal{Q}}$ which induces an isomorphism $\mathfrak{b} \to \mathfrak{u}$ is holomorphic.*

Definition 2.18 *A pair $(\hat{J}, \varphi)$ of type $(\hat{Q}, J_0, \hat{\alpha})$ is called σ-__polystable__ if:*

1. *its infinitesimal stabilizer $\mathfrak{g}_{\hat{J}, \varphi}$ is a reductive subalgebra of $A^0(\hat{Q} \times_{\mathrm{ad}} \mathfrak{g})$,*

2. *it is σ-semistable and the equality*

$$\frac{2\pi}{(n-1)!} \deg_g(\rho, h(\xi)) + \int_{X_\rho} \lambda^\xi(\varphi, \rho) vol_g = 0$$

   in the semistability condition holds when and only when ρ is induced by a global __holomorphic__ $Z_{\hat{G}}(\xi)$-reduction $\hat{\mathcal{Z}}(\xi) \hookrightarrow \hat{\mathcal{Q}}_{\hat{j}}$, and the section

$$\xi_{\hat{\mathcal{Z}}(\xi)} \in H^0(X, \hat{\mathcal{Q}}_{\hat{j}} \times_{\mathrm{ad}} \mathfrak{g})$$

   defined by ξ via this reduction belongs to the infinitesimal stabilizer $\mathfrak{g}_{\hat{J}, \varphi}$.

Definition 2.19 *Let $z(\mathfrak{g}, \hat{G}) \subset z(\mathfrak{g})$ be the centralizer of $\hat{G}$ in $\mathfrak{g}$, i. e. the subalgebra consisting of those elements of $\mathfrak{g}$ which are fixed under the adjoint $\hat{G}$-action. In other words, $z(\mathfrak{g}, \hat{G})$ is the Lie algebra of $Z(G, \hat{G}) := G \cap Z(\hat{G})$.*

When $\hat{G}$ is connected, one has $z(\mathfrak{g}, \hat{G}) = z(\mathfrak{g})$. Note that $z(\mathfrak{g}, \hat{G})$ can always be identified with a finite dimensional abelian subalgebra of $A^0(\hat{Q} \times_{\mathrm{ad}} \mathfrak{g})$. For pairs $(\hat{J}, \varphi)$ with infinitesimal stabilizer contained in $z(\mathfrak{g}, \hat{G})$ the polystability condition becomes:

Remark 2.20 *Let $\beta = (\hat{J}, \varphi)$ be a pair with infinitesimal stabilizer $\mathfrak{g}_\beta$ contained in $z(\mathfrak{g}, \hat{G})$. Then β is polystable if and only if it is semistable and the inequality in the semistability condition is strict unless $\xi \in H(G) \cap \mathfrak{g}_\beta$, in which case one must have $\hat{G}(\xi) = \hat{G}$, $\hat{\mathcal{Q}}_{\hat{j}}^\rho = \hat{\mathcal{Q}}_{\hat{j}}$.*

We show now that, by a suitable reduction of the structure group of the bundle, one can reduce the study of __any__ polystable pair $\beta = (\hat{J}, \varphi)$ to the case $\mathfrak{g}_\beta \subset z(\mathfrak{g}, \hat{G})$ (compare with Step 1 in the proof of Theorem 1.15 in the finite dimensional framework).

Let β be a σ-polystable pair, $\mathfrak{b} \subset \mathfrak{g}$ a reductive subalgebra and $\hat{Z} \subset \hat{Q}$ a $Z_{\hat{G}}(\mathfrak{b})$-reduction of $\hat{Q}$ such that the infinitesimal stabilizer $\mathfrak{g}_\beta$ is identified with $\mathfrak{b}$ via this reduction (see Definition 2.16). Put

$$\hat{G}' := Z_{\hat{G}}(\mathfrak{b}) \ , \ G' := Z_G(\mathfrak{b}) = \hat{G}' \cap G \ , \ G_0' := {}^{\hat{G}'}\!/_{G'} \subset G_0 \ , \ \hat{\alpha}' := \hat{\alpha}|_{\hat{G}' \times F} \ .$$

By Remark 2.17 it follows that $\hat{Z}$ is a holomorphic reduction of $\hat{\mathcal{Q}}_{\hat{j}}$; we denote by $\hat{J}'$ the induced holomorphic structure.

Put $Z_0 := \hat{Z} \times_{\hat{G}'} G_0'$, and note that Z_0 is a holomorphic G_0'-subbundle of $\mathcal{Q}_0$, so it has an induced holomorphic structure J_0'.

The restriction of the maximal weight function $\lambda : H(G) \times F \to \mathbb{R} \cup \{\infty\}$ to $H(G') \times F \to \mathbb{R} \cup \{\infty\}$ is the maximal weight function λ' of a suitable Hamiltonian triple (K', g', μ') for the restricted action $\alpha' := \alpha|_{F \times \hat{G}'}$, where μ' is equivariant with respect to a maximal compact subgroup $\hat{K}' \supset K'$ of $\hat{G}'$.

Proposition 2.21 *If $\beta = (\hat{J}, \varphi)$ is σ-polystable, then the associated pair $\beta' := (\hat{J}', \varphi)$ is a σ'-polystable pair of type $(\hat{Z}, J_0', \hat{\alpha}')$. The infinitesimal stabilizer $\mathfrak{g}_{\beta'}'$ of this pair is $\mathfrak{b} \cap z_{\mathfrak{g}}(\mathfrak{b}) = z(\mathfrak{b})$ which is contained in the centralizer $z(\mathfrak{g}', \hat{G}')$.*

Proof: Consider the holomorphic bundles $\hat{\mathcal{Q}} := \hat{\mathcal{Q}}_{\hat{J}}$, $\hat{\mathcal{Z}} := \hat{\mathcal{Z}}_{\hat{J}'}$ defined by the holomorphic structures $\hat{J}$, $\hat{J}'$. It is easy to see that $(\hat{J}', \varphi)$ is σ'-semistable because, for any $s \in H(G')$ any meromorphic $\hat{G}'(s)$-reduction ρ' of $\hat{\mathcal{Z}}$ extends to a meromorphic $\hat{G}(s)$-reduction ρ of $\hat{\mathcal{Q}}$, and the two total maximal weights coincide.

The delicate part is the fact that β' is σ'-polystable. Let ρ' be a meromorphic $\hat{G}'(s)$-reduction of $\hat{\mathcal{Z}}$ such that the corresponding total maximal weight vanishes, and let ρ be the associated $\hat{G}(s)$-reduction of $\hat{\mathcal{Q}}$. We know that ρ is induced by a global holomorphic $Z_{\hat{G}}(s)$-reduction $\mathcal{Z}(s) \subset \hat{\mathcal{Q}}$ such that the section $\sigma := s_{\hat{\mathcal{Z}}(s)}$ defined by s via this reduction belongs to the infinitesimal stabilizer $\mathfrak{g}_\beta$. We claim that, under these assumptions, one has

1. s coincides with the element $b \in \mathfrak{b}$ defined by the equality $\sigma = b_{\hat{\mathcal{Z}}}$.

2. $\hat{G}'(s) = \hat{G}'$, in particular the meromorphic reduction ρ' is just $\hat{\mathcal{Z}} \overset{=}{\hookrightarrow} \hat{\mathcal{Z}}$.

In order to prove this, note first that

$$\hat{\mathcal{Z}}(s) = \{q \in \hat{\mathcal{Q}} |\ \sigma(q) = s\} \ . \tag{16}$$

On the other hand, $\hat{\mathcal{Z}}(s)|_{X_\rho} \subset \hat{\mathcal{Q}}^\rho \subset \hat{\mathcal{Z}}|_{X_\rho} \cdot \hat{G}(s)$, because, by construction, $\hat{\mathcal{Q}}^\rho$ is the $\hat{G}(s)$-saturation of a $\hat{G}'(s)$-bundle contained in $\hat{\mathcal{Z}}$. Since X_ρ is dense, the inclusion $\hat{\mathcal{Z}}(s) \subset \hat{\mathcal{Z}} \cdot \hat{G}(s)$ holds on whole X. Therefore, any $q \in \hat{\mathcal{Z}}(s)$ has the form zg with $z \in \hat{\mathcal{Z}}$ and $g \in \hat{G}(s)$. We get, for any $q \in \hat{\mathcal{Z}}(s)$

$$s = \sigma(q) = \sigma(zg) = \mathrm{ad}_{g^{-1}}(\sigma(z)) = \mathrm{ad}_{g^{-1}}(b) \ ,$$

where $b \in \mathfrak{b}$ is the element which corresponds to $\sigma \in \mathfrak{g}_{\hat{J}, \varphi}$ via the reduction $\hat{\mathcal{Z}}$.

Therefore $b = \mathrm{ad}_g(s)$ with $g \in \hat{G}(s)$. Now recall that $\mathrm{ad}_{\hat{G}(s)}(s) = s + \mathfrak{u}(s)$. But $b \in z_{\hat{\mathfrak{g}}}(s)$ (because $s \in z_{\hat{\mathfrak{g}}}(\mathfrak{b})$). We get $b = s$. The second statement follows from the first. ∎

3 Hermitian-Einstein pairs

3.1 The Hermitian-Einstein equation

Choose a maximal compact subgroup K of G and a maximal compact subgroup $\hat{K}$ of $\hat{G}$ which contains K. Let $\hat{P} \subset \hat{Q}$ be a $\hat{K}$-reduction of $\hat{Q}$, and let P_0 be the bundle associated with $\hat{P}$ with structure group $K_0 := {}^{\hat{K}}\!/_K$. Consider the (unique) integrable connection $A_0 \in \mathcal{A}^{1,1}(P_0)$ which corresponds to J_0 via the Chern correspondence. Using the Chern correspondence (see section 6.1), our classification problem associated with the complex data $(\hat{Q}, J_0, \alpha)$ can be reformulated as follows:

Classify the pairs $(\hat{A}, \varphi)$, where $\hat{A}$ is an integrable connection on $\hat{P}$ which induces a fixed integrable connection A_0 on P_0, and φ is a J_A-holomorphic section in the associated bundle $E = \hat{P} \times_{\hat{K}} F$. The classification is considered up to isomorphy defined by the action of the complex gauge group

$$\mathcal{G} := \Gamma(X, \hat{P} \times_{\mathrm{Ad}_{\hat{K}}} G) = \mathrm{Aut}_{Q_0}(\hat{Q}) \ .$$

This action is induced – via the Chern correspondence – by the $\mathcal{G}$-action on the space of holomorphic pairs.

A pair $(\hat{A}, \varphi)$ as above will be called *an integrable pair of type* $(\hat{P}, A_0, \alpha)$.

Note that, formally, $\mathcal{G}$ can be regarded as the complexification of the real gauge group

$$\mathcal{K} := \Gamma(X, \hat{P} \times_{\mathrm{Ad}_{\hat{K}}} K) = \mathrm{Aut}_{P_0}(\hat{P}) \ .$$

We denote by $\mathcal{A}_{A_0}(\hat{P})$ the space of connections on $\hat{P}$ which induce A_0 on P_0.

Recall that we have fixed an ad-invariant complex inner product of Euclidean type h on $\hat{\mathfrak{g}}$ (see Definition 2.5). This inner product defines an orthogonal projection $\mathrm{pr}_{i\hat{\mathfrak{k}}} : i\hat{\mathfrak{k}} \to i\mathfrak{k}$ which induces a bundle projection

$$\hat{P} \times_{\mathrm{ad}} i\hat{\mathfrak{k}} \longrightarrow \hat{P} \times_{\mathrm{ad}} i\mathfrak{k}$$

which will be denoted by the same symbol.

Using the inner product defined by h to $\mathfrak{k}$, we can view μ as a map with values in $\mathfrak{k}$.

Definition 3.1 *An pair $(\hat{A}, \varphi)$ of type $(\hat{P}, A_0, \hat{\alpha})$ is called μ-Hermitian-Einstein if it is integrable and solves the equation*

$$\mathrm{pr}_{i\mathfrak{k}} \left[i\Lambda_g F_{\hat{A}} \right] + i\mu(\varphi) = 0 \ , \tag{HE}$$

which will be called the generalized Hermitian-Einstein equation associated with the data $(\hat{P}, A_0, \hat{\alpha}, \mu, g)$.

Let $(\hat{J}, \varphi)$ be a holomorphic pair of type $(\hat{Q}, J_0, \hat{\alpha})$. A $\hat{K}$-reduction $\hat{P}$ of $\hat{Q}$ will be called μ-Hermitian-Einstein if the associated pair $(A_{\hat{P},\hat{J}}, \varphi)$ is μ-Hermitian-Einstein.

Here we denoted by $A_{\hat{P},\hat{J}}$ the Chern connection of the pair $(\hat{P}, \hat{J})$ (see section 6.1).

Note that this equation is $\mathcal{G}$-invariant, so that one can consider the moduli space of solutions

$$\mathcal{M}^{\mathrm{HE}} := [\mathcal{A}_{A_0}(\hat{P}) \times \Gamma(E)]^{\mathrm{HE}} \big/ \mathcal{G} \ ,$$

where $[\mathcal{A}_{A_0}(\hat{P}) \times \Gamma(E)]^{\mathrm{HE}} \subset [\mathcal{A}_{A_0}(\hat{P}) \times \Gamma(E)]$ is the subspace of (integrable) pairs satisfying the equation (HE).

Note that any section $v \in \Gamma(X, \hat{Q} \times_{\mathrm{ad}} \mathfrak{g})$ induces a section $v^{\#}$ in the vertical tangent bundle V_E of E.

Definition 3.2 *A pair $(\hat{A}, \varphi)$ of type $(\hat{P}, A_0, \alpha)$ is called irreducible if its infinitesimal stabilizer*

$$\mathfrak{k}_{(\hat{A}, \varphi)} := \{ v \in A^0(\hat{P} \times_{\mathrm{ad}} \mathfrak{k}) | \ \nabla_{\hat{A}} v = 0, \ v^{\#} \circ \varphi = 0 \}$$

(which is a sub-Lie-algebra of $A^0(\hat{P} \times_{\mathrm{ad}} \mathfrak{k})$) vanishes.

3.2 Pairs which allow Hermitian-Einstein reductions are polystable

We come back to the assumptions and notations of section 2.2.2. The aim of this section is to prove the following

Theorem 3.3 *Let $(\hat{J}, \varphi)$ be a holomorphic pair of type $(\hat{Q}, J_0, \alpha)$. Suppose that there exists a $\hat{K}$-reduction $\hat{P} \subset \hat{Q}$ such that the associated pair $(A_{\hat{P},\hat{J}}, \varphi)$ is μ-Hermitian-Einstein. Then $(\hat{J}, \varphi)$ is σ-polystable, and it is stable if only if $(\hat{A}_{\hat{P},\hat{J}}, \varphi)$ is irreducible.*

Proof:

We will assume that g is Gauduchon (see Remark 2.12). We check first the inequality in the definition of σ-semistability. By Remark 2.15 it suffices to check this inequality for elements $\xi \in i\mathfrak{k}$.

Let ρ be a meromorphic reduction of $\hat{Q}$ to $\hat{G}(\xi)$, and put $\hat{P}^{\rho} := \hat{Q}^{\rho} \cap \hat{P}$. The inclusion $\iota : \hat{P}^{\rho} \hookrightarrow \hat{P}|_{X_{\rho}}$ defines a reduction of the bundle $\hat{P}|_{X_{\rho}}$ to the subgroup $\hat{K}(\xi) := \hat{G}(\xi) \cap \hat{K}$ of K. It is easy to see that

$$\hat{K}(\xi)^{\mathbb{C}} = Z_{\hat{G}}(\xi) .$$

Therefore ξ is invariant under the structure group of $\hat{P}^{\rho}$, so it defines a section $s(\rho, \xi)$ in $A^0(\hat{P}|_{X_{\rho}} \times_{\mathrm{ad}} i\mathfrak{k})$.

Put $\hat{A} := A_{\hat{P}, \hat{j}}$. We get

$$0 = \int_{X_{\rho}} \left\langle \mathrm{pr}_{i\mathfrak{k}} \left[i\Lambda_g F_{\hat{A}} \right] + i\mu(\varphi), s(\rho, \xi) \right\rangle vol_g =$$

$$= \int_{X_{\rho}} \left\langle i\Lambda_g F_{\hat{A}}, s(\rho, \xi) \right\rangle vol_g + \int_{X_{\rho}} i\mu(\varphi)(s(\rho, \xi)) vol_g .$$

Recall now that the functions $t \mapsto \mu_{\xi}(e^{t\xi})$ involved in the definition of λ are monotonely increasing, so that, in every point $x \in X_{\rho}$ we get

$$i\mu(\varphi)(s(\rho, \xi))(x) \leq \lambda^{s(\rho,\xi)}(\varphi)(x) .$$

This shows that

$$\int_{X_{\rho}} i\mu(\varphi)(s(\rho, \xi)) vol_g \leq \int_{X_{\rho}} \lambda^{s(\rho,\xi)}(\varphi) vol_g = \int_{X_{\rho}} \lambda^{\xi}(\varphi) vol_g ,$$

with equality if and only if $s(\rho, \xi)^{\#} \circ \varphi = 0$ over X_{ρ}.

The adjoint bundle $\hat{P} \times_{\mathrm{ad}} \hat{\mathfrak{k}} = \hat{P}^{\rho} \times_{\mathrm{ad}} \hat{\mathfrak{k}}$ of $\hat{P}$ splits as

$$\hat{P}^{\rho} \times_{\mathrm{ad}} \hat{\mathfrak{k}} = \mathrm{ad}(\hat{P}^{\rho}) \oplus \left[\hat{P}^{\rho} \times_{\mathrm{ad}} [\hat{\mathfrak{k}}(\xi)]^{\perp} \right] .$$

The connection $\hat{A}$ on $\hat{P}$ induces a connection $\hat{A}_{\rho}$ on $\hat{P}^{\rho}$ whose horizontal space at a point $p \in \hat{P}^{\rho}$ is the intersection

$$\left[H_{\hat{A}, p} \oplus p \cdot [\hat{\mathfrak{k}}(\xi)]^{\perp} \right] \cap T_p(\hat{P}^{\rho}) ,$$

where $H_{\hat{A}, p}$ stands here for the $\hat{A}$-horizontal space. The connection form of this connection is

$$\omega_{\hat{A}_{\rho}} = \mathrm{pr}_{\mathrm{ad}(\hat{P}^{\rho})} \iota^*(\omega_{\hat{A}})$$

The difference $a := \hat{A} - \iota_*(\hat{A}_{\rho})$ is a section of the subbundle $\hat{P}^{\rho} \times_{\mathrm{ad}} [\hat{\mathfrak{k}}(\xi)]^{\perp}$ of $\mathrm{ad}(\hat{P})$ (the second fundamental form of the subbundle $\hat{P}^{\rho}$ of $\hat{P}$ with respect to the connection $\hat{A}$).

Decompose the Lie algebra $\hat{\mathfrak{g}}$ as

$$\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_+ \oplus \hat{\mathfrak{g}}_- \oplus \hat{\mathfrak{g}}_0$$

where $\hat{\mathfrak{g}}_\pm$ is direct sum of the eigenspaces of the endomorphism

$$[\xi, \cdot] \in \mathrm{End}(\hat{\mathfrak{g}})$$

corresponding to the positive (negative) eigenvalues, and $\hat{\mathfrak{g}}_0 = \ker[\xi, \cdot]$. Note that

$$\hat{\mathfrak{g}}_0 = [\hat{\mathfrak{k}}(\xi)]^{\mathbb{C}}, \ \hat{\mathfrak{g}}_- \oplus \hat{\mathfrak{g}}_0 = \hat{\mathfrak{g}}(\xi), \ \hat{\mathfrak{g}}_+ = [\hat{\mathfrak{k}}(\xi)]^{\perp}, \ \hat{\mathfrak{g}}_+ = \overline{\hat{\mathfrak{g}}_-} \qquad (1)$$

where the conjugation in the last formula is taken with respect to the real structure $\hat{\mathfrak{g}} = i\hat{\mathfrak{k}} \otimes \mathbb{C}$ on $\hat{\mathfrak{g}}$.

Therefore, the second fundamental form a decomposes as

$$a = a^+ + a^-$$

where $a_\pm \in A^1(X, \hat{P}^\rho \times_{\mathrm{ad}} \mathfrak{g}_\pm)$ and

$$a^- = -\bar{a}^+ \ . \qquad (17)$$

But $\hat{\mathfrak{g}}_+ = T_{[e]}\left[\hat{G}\big/_{\hat{G}(\xi)} \right]$ and on X_ρ

$$\hat{P}^\rho \times_{\mathrm{ad}} \hat{\mathfrak{g}}_+ = \hat{Q}^\rho \times_{\mathrm{ad}} \hat{\mathfrak{g}}_+ = \rho^* \left(T^V \left(\hat{Q} \times_{\hat{G}} \left[\hat{G}\big/_{\hat{G}_\xi} \right] \right) \right) \ ,$$

(where T^V denotes as usually the vertical tangent bundle of a fibration). Therefore the holomorphy of ρ is equivalent to the condition

$$a^+ \in A^{10}(X, \hat{P}^\rho \times_{\mathrm{ad}} \hat{\mathfrak{g}}_+) \qquad (18)$$

(compare with Proposition 6.27). Write

$$a = \sum_{\lambda \in \mathrm{Spec}[\xi, \cdot]} a_\lambda$$

so that

$$a^\pm = \sum_{\lambda \in \mathrm{Spec}^\pm [\xi, \cdot]} a_\lambda \ ,$$

and take into account that

$$[\mathrm{Eig}([\xi, \cdot], \lambda), \mathrm{Eig}([\xi, \cdot], \mu)] \subset \mathrm{Eig}([\xi, \cdot], \lambda + \mu) \ ,$$

$$\overline{\mathrm{Eig}([\xi, \cdot], \lambda)} = \mathrm{Eig}([\xi, \cdot], -\lambda) \ .$$

By (17), (18) it follows that

$$\mathrm{pr}_{\hat{\mathfrak{k}}_\xi}[a \wedge a] = \sum_{\lambda \in \mathrm{Spec}^+[\xi,\cdot]} [a_\lambda \wedge a_{-\lambda}] + \sum_{\lambda \in \mathrm{Spec}^-[\xi,\cdot]} [a_\lambda \wedge a_{-\lambda}] =$$

$$= -2 \sum_{\lambda \in \mathrm{Spec}^+[\xi,\cdot]} [a_\lambda \wedge \overline{a}_\lambda] , \tag{19}$$

where $[\cdot \wedge \cdot]$ is the (symmetric) bilinear multiplication obtained by multiplying the Lie algebra bracket with the wedge product on 1-forms:

$$[u \wedge v](x,y) := [u(x), v(y)] - [u(y), v(x)] .$$

Comparing the curvatures of the two connections $\hat{A}$, $\hat{A}_\rho$, we get

$$\mathrm{pr}_{\hat{\mathfrak{k}}_\xi} F_{\hat{A}} = F_{\hat{A}_\rho} + \mathrm{pr}_{\hat{\mathfrak{k}}_\xi} \frac{1}{2}[a \wedge a] . \tag{20}$$

But, since $\langle \cdot, \cdot \rangle$ is ad-invariant, we get for any $u, v \in \hat{\mathfrak{g}}$ with $u \in \mathrm{Eig}([\xi, \cdot], \lambda)$

$$\langle [u, v], \xi \rangle^{\mathbb{C}} = -\langle v, [u, \xi] \rangle^{\mathbb{C}} = \langle [\xi, u], v \rangle^{\mathbb{C}} = \lambda \langle u, v \rangle^{\mathbb{C}} . \tag{21}$$

Therefore, by the degree formula given in Proposition 2.11, one obtains

$$\int_{X_\rho} \langle i\Lambda_g F_{\hat{A}}, s(\rho, \xi)) \rangle \, vol_g = \frac{2\pi}{(n-1)!} \mathrm{deg}(\rho, h(\xi)) - \sum_{\lambda \in \mathrm{Spec}^+[\xi,\cdot]} \lambda \parallel a_\lambda \parallel^2_{L^2} , \tag{22}$$

hence $\frac{2\pi}{(n-1)!} \mathrm{deg}(\rho, h(\xi)) \geq \int_{X_\rho} \langle i\Lambda_g F_{\hat{A}}, s(\rho, \xi) \rangle vol_g$ with equality if and only if $\alpha = 0$, i. e. if and only if the connection $\hat{A}$ reduces to a $\hat{K}(\xi)$-connection, which implies that $s(\rho, \xi)$ is an $\hat{A}$-parallel section of $\hat{P} \times_{\mathrm{ad}} i\mathfrak{k}$.

Therefore we have shown that $(\hat{A}, \varphi)$ is σ-semistable, and that, for $\xi \in i\mathfrak{k}$, one has

$$\frac{2\pi}{(n-1)!} \mathrm{deg}_g(\rho, h(\xi)) + \int_{X_\rho} \lambda^\xi(\varphi, \rho) vol_g = 0$$

if and only if ρ is induced by an $\hat{A}$-parallel $\hat{K}(\xi)$-reduction of $\hat{P}$ and the associated section $s(\rho, \xi) \in A^0(\hat{P} \times_{\mathrm{ad}} i\mathfrak{k})$ belongs to $i\mathfrak{k}_{\hat{A},\varphi}$, where $\mathfrak{k}_{\hat{A},\varphi} \subset A^0(\hat{P} \times_{\mathrm{ad}} \mathfrak{k})$ is the infinitesimal stabilizer algebra of the pair $(\hat{A}, \varphi)$ with respect to the action of the real gauge group $\mathcal{K} := \Gamma(\hat{P} \times_{\mathrm{Ad}} K)$. But an $\hat{A}$-parallel $\hat{K}(\xi)$-reduction of $\hat{P}$ defines a holomorphic $Z_{\hat{G}}(\xi)$-reduction of $\hat{\mathcal{Q}}_J$. Therefore, the last condition in the definition of polystability is verified.

Using the same method as in the proof of Proposition 1.3 [Te3] one can easily check, using the maximum principle for the operator $P = i\Lambda\bar{\partial}\partial$, that

$$\mathfrak{g}_{\hat{J},\varphi} = \mathfrak{k}_{\hat{A},\varphi} \otimes \mathbb{C}$$

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(compare with Theorem 2.2.1 in [LT]). Taking into account that the elements of $\mathfrak{k}_{\hat{A},\varphi}$ are $\hat{A}$-parallel sections, we get an $\hat{A}$-parallel reduction of $\hat{P}$ to $Z_{\hat{K}}(\mathfrak{c})$, where $\mathfrak{c} \subset \mathfrak{k}$ is a subalgebra of $\mathfrak{k}$ isomorphic to $\mathfrak{k}_{\hat{A},\varphi}$. This gives a holomorphic $Z_{\hat{G}}(\mathfrak{c} \otimes \mathbb{C})$-reduction of $\hat{\mathcal{Q}}_{\hat{J}}$, so that $\mathfrak{g}_{\hat{J},\varphi}$ is a reductive subalgebra of $A^0(\hat{Q} \times_{\mathrm{ad}} \hat{\mathfrak{g}})$ in the sense of Definition 2.16, which proves the first condition in the definition of polystability. ∎

Remark 3.4 *The proof of the theorem shows that, for a Hermitian-Einstein pair $(\hat{A}, \varphi)$, the following conditions are equivalent:*

1. *$(\hat{A}, \varphi)$ is irreducible.*

2. *The associated holomorphic pair $(\hat{J}, \varphi)$ is simple.*

4   Polystable pairs allow Hermitian–Einstein reductions

Definition 4.1 *We will denote by $\mathcal{R}(\hat{Q})$ the space of all $\hat{K}$-reductions of $\hat{Q}$ and, for a fixed K_0-reduction P_0 of Q_0, we will denote by $\mathcal{R}_{P_0}(\hat{Q})$ the space of all $\hat{K}$-reductions of $\hat{Q}$ which project onto P_0.*

Our final purpose is the following theorem:

Theorem 4.2 *Let $(\hat{J}, \varphi)$ be a σ-polystable pair of type $(\hat{Q}, \mathcal{Q}_0, \hat{\alpha})$. For any K_0-reduction P_0 of Q_0 there exists a reduction $\hat{P} \in \mathcal{R}_{P_0}(\hat{Q})$ which is μ- Hermitian-Einstein.*

In the proof, which will be completed at the and of section 4, we will assume again for simplicity that g is a Gauduchon metric; the statement is true for a general Hermitian metric (see Remark 2.12).

4.1   The perturbed equation

Let $(\hat{J}, \varphi)$ be a σ-polystable pair of type $(\hat{Q}, J_0)$, and let P_0 be a fixed K_0-reduction of Q_0. We seek a $\hat{K}$-reduction $\hat{P} \subset \hat{Q}$ which projects on P_0 such that the pair $(A_{\hat{P},\hat{J}}, \varphi)$ satisfies the HE equation (see Definition 3.1)), where $A_{\hat{P},\hat{J}}$ is the Chern connection of the pair $(\hat{P}, \hat{J})$.

By Proposition 2.21 we may suppose that

Assumption: *The infinitesimal stabilizer $\mathfrak{g}_\beta$ of the polystable pair $\beta = (\hat{J}, \varphi)$ is contained in the centralizer $z(\mathfrak{g}, \hat{G})$.*

This assumption has the following important consequence:

Remark 4.3 *If this assumption is satisfied, the infinitesimal stabilizer of β*

$$\mathfrak{k}_\beta := \mathfrak{g}_\beta \cap A^0(\hat{P} \times_{\mathrm{ad}} \mathfrak{k})$$

with respect to the action of the real gauge group $\mathcal{K} := \Gamma(X, \hat{P} \times_{\mathrm{Ad}} K)$ is independent of the $\hat{K}$-reduction $\hat{P}$, and it can be identified with a subalgebra of $z(\mathfrak{k}, \hat{K})$. Moreover, for any $\hat{K}$-reduction $\hat{P}$ of $\hat{Q}$, one has

$$\mathfrak{g}_\beta = [\mathfrak{g}_\beta \cap A^0(\hat{P}^0 \times_{\hat{K}} \mathfrak{k})] \otimes \mathbb{C} = \mathfrak{k}_\beta \otimes \mathbb{C} . \tag{23}$$

Indeed, one has

$$\mathfrak{g}_\beta \cap A^0(\hat{P}^0 \times_{\hat{K}} \mathfrak{k}) = \mathfrak{g}_\beta \cap z(\mathfrak{g}, \hat{G}) \cap A^0(\hat{P} \times_{\hat{K}} \mathfrak{k}) = \mathfrak{g}_\beta \cap z(\mathfrak{k}, \hat{K}) .$$

The connected component G_β^0 of e in the stabilizer

$$G_\beta \subset \mathcal{G} := \Gamma(\hat{Q} \times_{\mathrm{Ad}} G)$$

of β is a connected closed complex subgroup of $\mathcal{G}$ with Lie algebra $\mathfrak{g}_\beta \subset z(\mathfrak{g}, \hat{G})$, so it is contained in the connected component $Z(G, \hat{G})^0$ of e in the intersection $Z(G, \hat{G}) := G \cap Z(\hat{G})$.

Now it is easy to see that, for every connected closed complex subgroup H of a complex torus Θ, one has $\mathfrak{h} = [\mathfrak{h} \cap \mathfrak{t}] \otimes \mathbb{C}$, where $\mathfrak{t}$ is the Lie algebra of the *unique* maximal compact subgroup T of Θ.

Therefore, in our case one gets

$$\mathfrak{g}_\beta = [\mathfrak{g}_\beta \cap z(\mathfrak{k}, \hat{K})] \otimes \mathbb{C} = [\mathfrak{g}_\beta \cap A^0(\hat{P}^0 \times_{\hat{K}} \mathfrak{k})] \otimes \mathbb{C} .$$

$\blacksquare$

We fix a $\hat{K}$-reduction $\hat{P} \in \mathcal{R}_{P_0}(\hat{Q})$ and seek a solution $\hat{P} \subset \hat{Q}$ of our problem of the form

$$\hat{P}_s = e^{-\frac{s}{2}}(\hat{P}) ,$$

where $s \in A^0(\hat{P} \times_{\mathrm{ad}} i\mathfrak{k})$. Put $h := e^s$. The new (pointwise) moment map $\mu : \Gamma(\hat{Q} \times_{\hat{\alpha}} F) \to A^0(\hat{P}_s \times_{\mathrm{ad}} \mathfrak{k})$ associated with the reduction $\hat{P}_s$ is given by the formula

$$\mu_s(\varphi) = \mathrm{ad}_{e^{-\frac{s}{2}}}(\mu(e^{\frac{s}{2}}\varphi)) , \tag{24}$$

where μ is the moment map with respect to the initial reduction $\hat{P}$.

Denote by A^0 the Chern connection of the pair $(\hat{P}, \hat{J})$. The Hermitian-Einstein equation (Definition 3.1) becomes

$$\mathrm{pr}_{i\mathfrak{k}}\left[i\Lambda_g(F_{A^0} + \bar{\partial}(e^{-s}\partial_0(e^s)))\right] + i\mathrm{ad}_{e^{-\frac{s}{2}}}(\mu(e^{\frac{s}{2}}\varphi)) = 0 \ , \tag{25}$$

or, equivalently,

$$\mathrm{pr}_{i\mathfrak{k}} \, \mathrm{ad}_{e^{\frac{s}{2}}}\left[i\Lambda_g(F_{A^0} + \bar{\partial}(e^{-s}\partial_0(e^s)))\right] + i(\mu(e^{\frac{s}{2}}\varphi)) = 0 \ . \tag{26}$$

Our perturbed equation is

$$\mathrm{pr}_{i\mathfrak{k}}\left[i\Lambda_g(F_{A^0} + \bar{\partial}(e^{-s}\partial_0(e^s))]\right] + i\mathrm{ad}_{e^{-\frac{s}{2}}}(\mu(e^{\frac{s}{2}}\varphi)) + \varepsilon s = 0 \ , \tag{27}$$

which, since s is $\mathrm{ad}_{e^{\frac{s}{2}}}$ – invariant, is equivalent to

$$\mathrm{ad}_{e^{\frac{s}{2}}}\mathrm{pr}_{i\mathfrak{k}}\left[i\Lambda_g(F_{A^0} + \bar{\partial}(e^{-s}\partial_0(e^s)))\right] + i(\mu(e^{\frac{s}{2}}\varphi)) + \varepsilon s = 0 \ . \tag{28}$$

Proposition 4.4 *There exists a $\hat{K}$-reduction $\hat{P} \in \mathcal{R}_{P_0}(\hat{Q})$ such that, denoting by A^0 the corresponding Chern connection $\hat{A}_{\hat{P},\hat{j}}$, one has:*

A.
$$\mathrm{pr}_{A^0(X,i\mathfrak{k}_\beta)}\left[\mathrm{pr}_{i\mathfrak{k}}[i\Lambda_g F_{\hat{A}^0}] + i\mu(\varphi)\right] = 0 \tag{29}$$

B. Equation (27) has a solution $\zeta \in A^0(\hat{P}^0 \times_{\mathrm{ad}} i\mathfrak{k}_\beta^\perp)$ for $\varepsilon = 1$.

Proof:

Step 1. We start with an arbitrary reduction $\hat{P}^1 \in \mathcal{R}_{P_0}(\hat{Q})$, and we seek a smooth function $v \in \mathcal{C}^\infty(X, i\mathfrak{k}_\beta) \subset \mathcal{C}^\infty(X, iz(\mathfrak{k}))$ such that the Chern connection $\hat{A}^2 := A_{\hat{P}^2,\hat{j}}$ associated with the reduction $\hat{P}^2 := e^{-\frac{v}{2}}(\hat{P}^1)$ satisfies (29).

By Proposition 6.2, Corollary 6.3 one has

$$\hat{A}^2 := \hat{A}^1 + \partial v \ , \ F_{\hat{A}^2} := F_{\hat{A}^1} + \bar{\partial}\partial v \ .$$

The new moment map with respect to the reduction $\hat{P}^2$ is given by

$$\mu_2(\varphi) = \mathrm{ad}_{e^{-\frac{v}{2}}} \ \mu_1(e^{\frac{v}{2}}\varphi) = \mu_1(\varphi) \ ,$$

because v is central and belongs pointwise to the infinitesimal stabilizer of φ. Therefore (29) becomes

$$\mathrm{pr}_{A^0(X,i\mathfrak{k}_\beta)}\left[\mathrm{pr}_{i\mathfrak{k}}[i\Lambda_g F_{\hat{A}^1}] + i\mu_1(\varphi)\right] + i\Lambda\bar{\partial}\partial v = 0 \ . \tag{30}$$

On the other hand, applying the polystability condition to central elements of the form $u \in i\mathfrak{k}_\beta$ one gets that

$$\langle \mathrm{pr}_{i\mathfrak{k}}[i\Lambda_g F_{\hat{A}^1}] + i\mu_1(\varphi), u \rangle_{L^2} = 0 \ \forall u \in i\mathfrak{k}_\beta \tag{31}$$

(compare with Remark 1.12 in the finite dimensional framework). This implies that the Laplace equation (30) has solutions, so we get a reduction $\hat{P}2$ with the desired property.

Step 2.

Put

$$\zeta := -\mathrm{pr}_{i\mathfrak{k}}[i\Lambda_g F_{\hat{A}^2}] - i\mu_2(\varphi)$$

and $\hat{P} := e^{\frac{\zeta}{2}}(\hat{P}2)$. By Step 1 it follows that $\zeta \in A^0(\hat{P}2 \times_{\mathrm{ad}} i\mathfrak{k}_\beta^\perp)$. Since $e^{\frac{\zeta}{2}}$ commutes with ζ, one can also regard ζ as an element of $A^0(\hat{P} \times_{\mathrm{ad}} i\mathfrak{k})$ which will be also pointwise orthogonal to $\mathfrak{k}_\beta$. By Corollary 6.3 and (24) one has

$$-\zeta = \mathrm{pr}_{i\mathfrak{k}}\left[i\Lambda_g(F_{A^2}] + i(\mu_2(\varphi)) = \right.$$

$$= \mathrm{pr}_{i\mathfrak{k}}\left[i\Lambda_g(F_{A^0} + \bar{\partial}(e^{-\zeta}\partial_0(e^\zeta)))\right] + i\mathrm{ad}_{e^{-\frac{\zeta}{2}}}(\mu_0(e^{\frac{\zeta}{2}}\varphi)) \ ,$$

which shows that, with these choices, B holds. ∎

4.2   A priori estimates for the solution s_ε

Lemma 4.5 *Fix a $\hat{K}$-reduction $\hat{P}$ as in Proposition 4.4. Let*

$$s \in A^0(\hat{P} \times_{\mathrm{ad}} i\mathfrak{k}_\beta^\perp)$$

be a section solving the equation

$$\mathrm{ad}_{e^{\frac{s}{2}}}\mathrm{pr}_{i\mathfrak{k}}\left[i\Lambda_g(F_{A^0} + \bar{\partial}(e^{-s}\partial_0(e^s)))\right] + i\mu(e^{\frac{s}{2}}\varphi) + \varepsilon s = 0 \ . \tag{e_ε}$$

Put

$$k^0 := \mathrm{pr}_{i\mathfrak{k}}\left[i\Lambda_g(F_{A^0}] + i\mu(\varphi) \ .$$

Then one has:

1. $\frac{1}{2} \cdot P(|s|^2) + \varepsilon \cdot |s|^2 \leq |k^0| \cdot |s|$;

2. $m := \sup_X |s| \leq \frac{1}{\varepsilon} \cdot \sup_X |k^0|$;

3. $m \leq C \cdot (\|s\|_{L^2} + \sup_X |k^0|)^2$ *where the constant C only depends on g and the fixed $\hat{K}$-reduction $\hat{P}$.*

Proof: Taking the pointwise inner product with respect to $\hat{P}$ of both sides of (e_ε) with s, one gets

$$(\text{pr}_{i\mathfrak{k}}[i\Lambda_g F_{A^0}], s) + \left(i\Lambda_g\bar{\partial}(e^{-s}\partial_0(e^s))), s\right) + 2\mu^{-i\frac{s}{2}}(e^{\frac{s}{2}}\varphi) + \varepsilon|s|^2 = 0 \ . \qquad (32)$$

On the other hand, note that

$$\mu^{-i\frac{s}{2}}(e^{\frac{s}{2}}\varphi) = \mu^{-i\frac{s}{2}}(\varphi) + \mathfrak{e}^{\frac{s}{2}}(\varphi) \ ,$$

where $\mathfrak{e}^s(\varphi)$ is the $\underline{\text{real}}$ $\underline{\text{analytic}}$-valued function which maps every $x \in X$ to the energy of the curve $[0,1] \ni t \mapsto e^{\frac{ts}{2}}\varphi(x) \in F_x$.

Therefore,

$$\left(i\Lambda_g\bar{\partial}(e^{-s}\partial_0(e^s))), s\right) + \varepsilon|s|^2 = -\left(\text{pr}_{i\mathfrak{k}}[i\Lambda_g(F_{A^0}], s) - 2\mu^{-i\frac{s}{2}}(e^{\frac{s}{2}}\varphi) \leq$$

$$- (\text{pr}_{i\mathfrak{k}}[i\Lambda_g F_{A^0}], s) - 2\mu^{-i\frac{s}{2}}(\varphi) = -(\text{pr}_{i\mathfrak{k}}[i\Lambda_g F_{A^0}] + i\mu(\varphi), s) = -(k^0, s) \ . \quad (33)$$

We introduce now the $\underline{\text{real}}$ $\underline{\text{analytic}}$ function $\eta : \mathbb{R} \to \mathbb{R}_{>0}$ defined by

$$\eta(t) = \left\{ \begin{array}{cc} \sqrt{\frac{1-e^{-t}}{t}} & \text{if} \quad t \neq 0 \\ 1 & \text{if} \quad t = 0 \end{array} \right.$$

(see section 6.4). We use the formalism explained in section 6.4. By Proposition 6.16 the inequality (33) becomes

$$\frac{1}{2}P(|s|^2) + |\eta([s,\cdot])(\partial_0(s))|^2 + \varepsilon|s|^2 \leq -(k^0, s) \ . \qquad (34)$$

This proves the claim.

2. follows from 1. by the Maximum Principle.

3. From 1. we deduce

$$P(|s|^2) \leq 2 \cdot |k^0||s| \leq |s|^2 + |k^0|^2 \leq |s|^2 + \sup_X |k^0|^2 \ ,$$

so, since $\sup_X |k^0|^2$ is a constant, Lemma 3.3.2 [LT] implies

$$\begin{aligned} m \quad &\leq C \cdot \left(\| \ |s|^2 \|_{L^1} + \sup_X |k^0|^2\right) = C \cdot \left(\|s\|_{L^2}^2 + \sup_X |s^0|^2\right) \\ &\leq C \cdot \left(\|s\|_{L^2} + \sup_X |k^0|\right)^2 \ . \end{aligned}$$

∎

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4.3   Solving the equation (e_ε) for $\varepsilon \in (0, 1]$.

Let
$$l(\varepsilon, s) := \mathrm{ad}_{e^{\frac{s}{2}}} \mathrm{pr}_{i\mathfrak{k}} \left[i\Lambda_g(F_{A^0} + \bar{\partial}(e^{-s}\partial_0(e^s))) \right] + i\mu(e^{\frac{s}{2}}\varphi) + \varepsilon s$$
be the left hand term of the equation (e_ε).

We state first the following regularity result: Choose $p \in \mathbb{N}$ such that the Sobolev space L_k^p is an algebra for $k \geq 1$.

Lemma 4.6 *Let* $s \in L_2^p(\hat{P} \times_{\mathrm{ad}} i\mathfrak{k}_\beta^\perp)$ *such that* $l(\varepsilon, s) = 0$ *for some* $\varepsilon \in \mathbb{R}$. *Then* $s \in \mathcal{C}^\infty(\hat{P} \times_{\mathrm{ad}} i\mathfrak{k}_\beta^\perp)$.

Proof: Recall first that, by Proposition 6.4, one has

$$\mathrm{ad}_{e^{\frac{s}{2}}}(F_{A^0} + \bar{\partial}(e^{-s}\partial_0(e^s))) = F_{A_s} , \tag{35}$$

where $A_s := e^{-\frac{s}{2}}(A^0 + e^{-s}\partial_0(e^s)) \in \mathcal{A}(\hat{P})$. The components of the linear connection defined by A_s on the bundle $\hat{P} \times_{\mathrm{ad}} \mathfrak{g}$ are

$$\bar{\partial}_{A_s} = \mathrm{ad}_{e^{\frac{s}{2}}} \circ \bar{\partial}_{A^0} \circ \mathrm{ad}_{e^{-\frac{s}{2}}} , \ \partial_{A_s} = \mathrm{ad}_{e^{-\frac{s}{2}}} \circ \partial_{A^0} \circ \mathrm{ad}_{e^{\frac{s}{2}}} , \tag{36}$$

which can be rewritten as

$$\bar{\partial}_{A_s} = \bar{\partial} - \bar{\partial}(e^{\frac{s}{2}})e^{-\frac{s}{2}} , \ \partial_{A_s} = \partial_0 + e^{-\frac{s}{2}}\partial_0(e^{\frac{s}{2}}) .$$

Here the forms $\bar{\partial}(e^{\frac{s}{2}})e^{-\frac{s}{2}}$, $e^{-\frac{s}{2}}\partial_0(e^{\frac{s}{2}})$ operate on $\hat{P} \times_{\mathrm{ad}} \mathfrak{g}$ via the adjoint representation $a \mapsto [a, \cdot]$. On the other hand, by formulae (88), (89) in the Appendix, these forms can be expressed in terms of $\bar{\partial}s, \partial_0 s$ as

$$\bar{\partial}(e^{\frac{s}{2}})e^{-\frac{s}{2}} = \frac{1}{2}\psi\left(-\left[\frac{s}{2}, \cdot\right]\right)(\bar{\partial}s) , \ e^{-\frac{s}{2}}\partial_0(e^{\frac{s}{2}}) = \frac{1}{2}\psi\left(\left[\frac{s}{2}, \cdot\right]\right)(\partial_0 s) , \tag{37}$$

where ψ is the positive real analytic function

$$\psi(t) = \begin{cases} \frac{e^t - 1}{t} & \text{if} \quad t \neq 0 \\ 1 & \text{if} \quad t = 0 . \end{cases}$$

Let $\Psi : \hat{P} \times_{\mathrm{ad}} i\mathfrak{k} \to \mathrm{Herm}(\hat{P} \times_{\mathrm{ad}} \mathfrak{g})$ be the (non-linear!) real analytic bundle map defined by

$$a \mapsto \frac{1}{2}\psi\left(\left[\frac{a}{2}, \cdot\right]\right) .$$

We get
$$\bar{\partial}_{A_s} = \bar{\partial} - \Psi(-s)(\bar{\partial}s) , \ \partial_{A_s} = \partial_0 + \Psi(s)(\partial_0 s) . \tag{38}$$
This gives for the $(1, 1)$ curvature form of A_s

$$F_{A_s} = F_{A^0} + d_{A^0}\left(\Psi(s)(\partial_0 s) - \Psi(-s)(\bar{\partial}s)\right) +$$

$$+ \big(\Psi(s)(\partial_0 s) - \Psi(-s)(\bar\partial s)\big) \wedge \big(\Psi(s)(\partial_0 s) - \Psi(-s)(\bar\partial s)\big) =$$

$$= 2\Psi(s)(\bar\partial\partial_0 s) + (1 - \Psi(-s))(F_{A^0}) + \bar\partial(\Psi(s)) \wedge \partial_0 s - \partial_0(\Psi(-s)) \wedge \bar\partial s$$

$$-\Psi(s)(\partial_0 s) \wedge \Psi(-s)(\bar\partial s) - \Psi(-s)(\bar\partial s) \wedge \Psi(s)(\partial_0 s) . \qquad (39)$$

Therefore the equation (e_ε) is equivalent to an equation of the form

$$i\Lambda\bar\partial\partial_0 s = \mathcal{F}(\varepsilon, s, d_{A^0}(s)) .$$

The left composition lemma applies on the right, more precisely, $\mathcal{F}(\varepsilon, s, d_{A^0}(s))$ belongs to L^p_{k-1} when s belongs to L^p_k. The result follows by standard bootstrapping. $\blacksquare$

The left composition lemma gives also the following

Remark 4.7 *The map l extends to a differentiable map*

$$l : \mathbb{R} \times L^p_2(\hat P \times_{\mathrm{ad}} i\mathfrak{k}^\perp_\beta) \to L^p(\hat P \times_{\mathrm{ad}} i\mathfrak{k}^\perp_\beta) .$$

Lemma 4.8 *Suppose that $l(\varepsilon, s) = 0$ with $s \in A^0(\hat P \times_{\mathrm{ad}} i\mathfrak{k}^\perp_\beta)$ and $\varepsilon > 0$. Then the partial derivative*

$$\frac{\partial}{\partial s} l(e, s) : L^p_2(\hat P \times_{\mathrm{ad}} i\mathfrak{k}^\perp_\beta) \to L^p(\hat P \times_{\mathrm{ad}} i\mathfrak{k}^\perp_\beta)$$

is an invertible operator.

Proof:

For $\dot s \in L^p_2(\hat P \times_{\mathrm{ad}} i\mathfrak{k}^\perp_\beta)$, put

$$u := (d_{\frac{s}{2}} \exp)(\frac{\dot s}{2}) \, e^{-\frac{s}{2}} \in \hat P \times_{\mathrm{ad}} (i\mathfrak{k}^\perp_\beta \otimes \mathbb{C}) .$$

The components of u in $\hat P \times_{\mathrm{ad}} i\mathfrak{k}^\perp_\beta$ and $\hat P \times_{\mathrm{ad}} \mathfrak{k}^\perp_\beta$ are respectively

$$u_h = \frac{1}{2}(u + \mathrm{ad}_{e^{-\frac{s}{2}}} u) , \ \ u_a = \frac{1}{2}(u - \mathrm{ad}_{e^{-\frac{s}{2}}} u) .$$

One has

$$(\frac{\partial}{\partial s} F_{A_s})(\dot s) = d_{A_s}(\frac{\partial}{\partial s} A_s) = d_{A_s}\big(\partial_{A_s}(\mathrm{ad}_{e^{-\frac{s}{2}}} u) - \bar\partial_{A_s} u\big) =$$

$$-\partial_{A_s}\bar\partial_{A_s} u + \bar\partial_{A_s}\partial_{A_s}(\mathrm{ad}_{e^{-\frac{s}{2}}} u) = [u_a, F_{A_s}] + (\bar\partial_{A_s}\partial_{A_s} - \partial_{A_s}\bar\partial_{A_s})(u_h) .$$

On the other hand, the same computation as in the finite dimensional framework (see the proof of Proposition 1.15) give

$$\frac{\partial}{\partial s} i\mu(e^{\frac{s}{2}}\varphi) = d(i\mu)(u^{\#}_{e^{\frac{s}{2}}\varphi}) = d(i\mu)([u^{\#}_h]_{e^{\frac{s}{2}}\varphi} + [u^{\#}_a]_{e^{\frac{s}{2}}\varphi}) =$$

$$= d(i\mu)([u^{\#}_h]_{e^{\frac{s}{2}}\varphi}) + [u_a, i\mu(e^{\frac{s}{2}}\varphi)] \ .$$

Therefore

$$\frac{\partial}{\partial s} l(\varepsilon, s)(\dot{s}) =$$

$$i\Lambda_g(\bar{\partial}_{A_s}\partial_{A_s} - \partial_{A_s}\bar{\partial}_{A_s})(u_h) + d(i\mu)([u^{\#}_h]_{e^{\frac{s}{2}}\varphi}) + [u_a, l(\varepsilon, s) - \varepsilon s] + \varepsilon\dot{s} \ . \quad (40)$$

Recall the following well-known identity which holds for any unitary integrable connection B on a Hermitian vector bundle:

$$P(|v|^2) := i\Lambda_g\bar{\partial}\partial|v|^2 = (i\Lambda_g\bar{\partial}_B\partial_B v, v) - (v, i\Lambda_g\partial_B\bar{\partial}_B v) - |d_B(v)|^2 \ .$$

In our case, using the Euclidean inner product on $\hat{P} \times_{\mathrm{ad}} i\mathfrak{k}$, this yields

$$P|u_h|^2 = (i\Lambda_g(\bar{\partial}_{A_s}\partial_{A_s} - \partial_{A_s}\bar{\partial}_{A_s})u_h, u_h) - |d_{A_s}(u_h)|^2 \ . \quad (41)$$

Therefore, for a pair (ε, s) with $l(\varepsilon, s) = 0$, and using the well known formulae obtained in the finite dimensional case, we get

$$\left\langle \frac{\partial}{\partial s} l(\varepsilon, s)(\dot{s}), u_h \right\rangle_{L^2} = \|d_{A_s} u_h\|^2_{L^2} + \|[u^{\#}_h]_{e^{\frac{s}{2}}\varphi}\|^2 - \varepsilon\langle[u_a, s], u_h\rangle + 2\varepsilon\langle\frac{\dot{s}}{2}, u_h\rangle$$

$$\geq \frac{1}{2}\varepsilon\|\dot{s}\|^2_{L^2} \quad (42)$$

by Proposition 6.10 in the Appendix. This proves that $\frac{\partial}{\partial s} l(\varepsilon, s)$ is an injective operator. It suffices to notice that its index vanishes. ∎

Lemma 4.9 *There is a positive constant $C = C(A^0, \varphi)$ such that the following estimate holds:*

$$C\|v\|^2_{L^2} \leq \|\bar{\partial}_{A^0} v\|^2_{L^2} + \|v^{\#}_{\varphi}\|^2_{L^2} \ \forall v \in A^0(\hat{P} \times_{\mathrm{ad}} (i\mathfrak{k}^{\perp}_{\beta} \otimes \mathbb{C})) \ .$$

Proof: If not, there would exist a sequence $(v_n)_n$ with $\|v_n\|^2_{L^2} \equiv 1$ such that

$$\|\bar{\partial}_{A^0} v_n\|^2_{L^2} + \|(v_n)^{\#}_{\varphi}\|^2_{L^2} \to 0$$

But this implies that v_n has a subsequence which is weakly convergent in L_1^2 and strongly convergent in L^2. The limit, say v_0, satisfies $\bar{\partial}_{A^0}(v_0) = 0$, hence it is smooth. Noting that $(\cdot)_\varphi^\#$ can be regarded as a smooth linear bundle morphism

$$\hat{P} \times_{\mathrm{ad}} (i\mathfrak{k}_\beta^\perp \otimes \mathbb{C})) \longrightarrow \varphi^*(T^{\mathrm{vert}}(E)) \ ,$$

one gets

$$\bar{\partial}_{A^0} v_0 = 0 \ , \ (v_0)_\varphi^\# = 0 \ .$$

Therefore, by Remark 4.3, one has $v_0 \in \mathfrak{g}_\beta = \mathfrak{k}_\beta \otimes \mathbb{C}$. But this contradicts $v_0 \in A^0(\hat{P} \times_{\mathrm{ad}} (i\mathfrak{k}_\beta^\perp \otimes \mathbb{C}))$. ∎

Note that this lemma holds for any holomorphic pair β and $\hat{K}$-reduction $\hat{P}$ such that the relation $\mathfrak{g}_\beta = \mathfrak{k}_\beta \otimes \mathbb{C}$ between its infinitesimal stabilizers holds.

For any $s \in A^0(\hat{P} \times_{\mathrm{ad}} (i\mathfrak{k}_\beta^\perp))$ put

$$m_s := \sup_X |s| \ .$$

Corollary 4.10 *There exists a positive constant $C(A^0, \varphi, m_s)$ such that for all $s \in A^0(\hat{P} \times_{\mathrm{ad}} (i\mathfrak{k}_\beta^\perp))$ and $w \in A^0(\hat{P} \times_{\mathrm{ad}} (i\mathfrak{k}_\beta^\perp \otimes \mathbb{C}))$ it holds*

$$C(A^0, \varphi, m_s) \|w\|_{L^2}^2 \leq \|\bar{\partial}_{A_s} w\|_{L^2}^2 + \|[w]_{e^{\frac{s}{2}}\varphi}^\#\|^2 \ .$$

Proof: Defining $v := \mathrm{ad}_{e^{-\frac{s}{2}}} w$ one has

$$\bar{\partial}_{A_s} w = \mathrm{ad}_{e^{\frac{s}{2}}}(\bar{\partial}_{A^0} v) \ , \ [w]_{e^{\frac{s}{2}}\varphi}^\# = (e^{\frac{s}{2}})_{*,\varphi}(v_\varphi^\#) \ .$$

But one has estimates of the form

$$\|\mathrm{ad}_{e^{\frac{s}{2}}}(\bar{\partial}_{A^0} v)\|_{L^2}^2 \geq C'(m_s)\|(\bar{\partial}_{A^0} v)\|_{L^2}^2 \ , \ \|(e^{\frac{s}{2}})_{*,\varphi}(v_\varphi^\#)\|_{L^2}^2 \geq C''(\varphi, m_s)\|(v_\varphi^\#)\|_{L^2}^2$$

$$\|v\|_{L^2}^2 \geq C'''(m_s)\|w\|_{L^2}^2 \ .$$

It suffices to apply the previous lemma. ∎

Let $(s_\varepsilon)_\varepsilon$ be a smooth family in $A^0(\hat{P} \times_{\mathrm{ad}} i\mathfrak{k}_\beta^\perp)$ such that for all ε it holds $l(\varepsilon, s_\varepsilon) = 0$. Put

$$u = (\frac{d}{d\varepsilon}\exp(\frac{s}{2}))e^{-\frac{s}{2}} \ , \ \dot{s} := \frac{d}{d\varepsilon}s \ .$$

Lemma 4.11 *One has an estimate of the form*

$$\sup_{X} |u_h| \leq C(m_s) . \tag{43}$$

Proof: By (40) one gets

$$0 = \frac{\partial}{\partial \varepsilon} l(\varepsilon, s_\varepsilon) =$$

$$i\Lambda_g(\bar{\partial}_{A_s}\partial_{A_s} - \partial_{A_s}\bar{\partial}_{A_s})(u_h) + d(i\mu)([u_h^\#]_{e^{\frac{s}{2}}\varphi}) - \varepsilon[u_a, s] + \varepsilon\dot{s} + s . \tag{44}$$

Taking this time inner product with u_h pointwise and using again (41) and the inequalities in Proposition 6.10, one gets

$$0 = P(|u_h|^2) + |d_{A_s}u_h|^2 + |[u_h^\#]_{e^{\frac{s}{2}}\varphi}|^2 - \varepsilon([u_a, s], u_h) + \varepsilon(\dot{s}, u_h)(\dot{s}, u_h) + (s, u_h) \geq$$

$$P(|u_h|^2) + |d_{A_s}u_h|^2 + |[u_h^\#]_{e^{\frac{s}{2}}\varphi}|^2 + (s, u_h) . \tag{45}$$

Integrating over X one gets

$$\|d_{A_s}u_h\|_{L^2}^2 + \|[u_h^\#]_{e^{\frac{s}{2}}\varphi}\|_{L^2}^2 \leq \|s\|_{L^2}\|u_h\|_{L^2} \leq \sqrt{Vol_g(X)}\, m_s\|u_h\|_{L^2} ,$$

whereas the term on the left is larger than $C(A^0, \varphi, m_s)\|u_h\|_{L^2}^2$ by Corollary 4.10. This gives an L^2-estimate of the form

$$\|u_h\|_{L^2} \leq C_1(A_0, \varphi, m_s) . \tag{46}$$

Coming back to (45) we see that

$$P(|u_h|^2) \leq -(s, u_h) \leq |s||u_h| \leq \frac{1}{2}(|u_h|^2 + m_s^2) .$$

Using now Lemma 3.3.2 in [LT] and (46), one gets an estimate of the form

$$\sup_{X}(|u_h|^2) \leq C(g)(\|u_h\|_{L^2}^2 + \frac{1}{2}m_s^2) \leq C(g, A^0, \varphi, m_s) . \tag{47}$$

$\blacksquare$

Using again the formulae (38) we get

$$\bar{\partial}_{A_s}\partial_{A_s} - \partial_{A_s}\bar{\partial}_{A_s} = \bar{\partial}\partial_0 - \partial_0\bar{\partial}$$

$$-\Psi(-s)(\bar{\partial}s)\wedge\partial_0 - \Psi(-s)(\bar{\partial}s)\wedge\Psi(s)(\partial_0 s) - \Psi(s)(\partial_0 s)\wedge\bar{\partial} + \Psi(s)(\partial_0 s)\wedge\Psi(-s)(\bar{\partial}s)$$

$$+\Psi(s)(\bar{\partial}\partial_0 s) + \Psi(-s)(\partial_0\bar{\partial}s) + \partial_0(\Psi(-s))\wedge\bar{\partial}s + \bar{\partial}(\Psi(s))\wedge\partial_0 s . \tag{48}$$

We plug the identity (48) in (44) and use standard elliptic L^p - estimates for the operator $i\Lambda_g(\bar{\partial}\partial_0 - \partial_0\bar{\partial})$.

Note first that, since Ψ is of class $\mathcal{C}^1$, one has an estimate of the form

$$|d_{A^0}(\Psi(s))| \leq C(m_s)|d_{A^0}s| \ .$$

Recalling that, by (43), $\sup_X |u_h|$ is bounded by a constant $C(m_s)$, we obtain

$$\|u_h\|_{L^p_2} \leq c(m_s)(1 + \|s\|_{L^{2p}_1}\|u_h\|_{L^{2p}_1} + \|s\|^2_{L^{2p}_1} + \|s\|_{L^p_2}) +$$

$$+ \|d(i\mu)([u_h^\#]_{e^{\frac{s}{2}}\varphi}) - \varepsilon[u_a, s] + \varepsilon\dot{s} + s\|_{L^p} \ . \tag{49}$$

On the other hand, by the formulae obtained in the proof of Proposition 6.10, one can express $\dot{s}$ in terms of u_h as

$$\dot{s} = 2\theta\left(\left[\frac{s}{2}, \cdot\right]\right)(u_h) \ ,$$

where this time $\theta(t) = \frac{2t}{e^t - e^{-t}}$. Since the bundle map $\hat{P} \times_{\mathrm{ad}} i\mathfrak{k} \to \mathrm{Herm}(\hat{P} \times_{\mathrm{ad}} \mathfrak{g})$ given by $a \mapsto 2\theta\left(\left[\frac{a}{2}, \cdot\right]\right)$ is of class $\mathcal{C}^2$, one gets estimates of the form

$$|\dot{s}| \leq c(m_s)|u_h| \ , \quad \|\dot{s}\|_{L^p_2} \leq c(m_s)\|u_h\|_{L^p_2} \ ,$$

in particular, from (43) we deduce

$$\sup_X |\dot{s}| \leq c(m_s) \ . \tag{50}$$

Expressing u_h and u_a similarly in terms of $\dot{s}$, one gets

$$|u_a| \leq c(m_s)|\dot{s}| \ , \quad \|u_h\|_{L^{2p}_1} \leq c(m_s)\|\dot{s}\|_{L^{2p}_1} \ .$$

Note finally that (since u_h and s are bounded in terms of m_s) $[u_h^\#]_{e^{\frac{s}{2}}\varphi}$ remains in a compact subset $K(\varphi, m_s)$ of the vertical tangent bundle $T^{\mathrm{vert}}(E)$. Therefore, one gets

$$\sup_X |d(i\mu)([u_h^\#]_{e^{\frac{s}{2}}\varphi})| \leq c(m_s) \ .$$

Summarizing, we obtain an estimate of the form

$$\|\dot{s}\|_{L^p_2} \leq c(m_s)\left(1 + \|s\|_{L^{2p}_1}\|\dot{s}\|_{L^{2p}_1} + \|s\|^2_{L^{2p}_1} + \|s\|_{L^p_2}\right) \ . \tag{51}$$

Using the general inequality

$$\|x\|_{L^{2p}_1} \leq C \cdot \left(\sup_X |x|^{\frac{1}{2}}\right) \cdot \|x\|^{\frac{1}{2}}_{L^p_2} + \|x\|_{L^{2p}} \ , \tag{52}$$

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([Au] Theorem 3.69; the constant C depends only on p and the dimension of X) and (50) one proceeds as in the proof of [LT] Proposition 3.3.5 to obtain an inequality of the form

$$\|\dot{s}\|_{L_2^p} \leq c(m_s)(1 + \|s\|_{L_2^p}) \; . \tag{53}$$

Integrating from ε to 1 this yields (compare with [LT], Proposition 3.3.5)

Proposition 4.12 *Let* $(s_\varepsilon)_{\varepsilon \in (\varepsilon_0, 1]}$ *be a smooth family in* $A^0(\hat{P} \times_{\mathrm{ad}} i\mathfrak{k}_\beta^\perp)$ *such that for all* ε *it holds* $l(\varepsilon, s_\varepsilon) = 0$. *Then one has an estimate of the form*

$$\|s_\varepsilon\|_{L_2^p} \leq e^{c(m_{s_\varepsilon})(1-\varepsilon)}(1 + \|s_1\|_{L_2^p}) \; \forall \varepsilon \in (\varepsilon_0, 1] \; . \tag{54}$$

We know that the equation $l(\cdot, 1) = 0$ has a smooth solution ζ (Proposition 4.4). Consider the set

$$\mathcal{S} := \left\{ \mathfrak{s} : (\varepsilon_\mathfrak{s}, 1] \to A^0(\hat{P} \times_{\mathrm{ad}} i\mathfrak{k}_\beta^\perp) \middle| \; 0 \leq \varepsilon_\mathfrak{s} < 1, \; l(\varepsilon, \mathfrak{s}(\varepsilon)) \equiv 0 \; , \; \mathfrak{s}(1) = \zeta \right\} \; .$$

$\mathcal{S}$ is nonempty by Lemma 4.8, implicit function theorem and Lemma 4.6. Writing $\mathfrak{s}_1 \leq \mathfrak{s}_2$ when $\varepsilon_{\mathfrak{s}_2} \leq \varepsilon_{\mathfrak{s}_1}$ and $\mathfrak{s}_1 = \mathfrak{s}_2|_{(\varepsilon_{\mathfrak{s}_1}, 1]}$, $\mathcal{S}$ becomes an inductively ordered set, hence there is a maximal element in $\mathcal{S}$.

Proposition 4.13
1. Let $\mathfrak{s} : (\varepsilon_\mathfrak{s}, 1] \to A^0(\hat{P} \times_{\mathrm{ad}} i\mathfrak{k}_\beta^\perp) \in \mathcal{S}$ and put $s_\varepsilon := \mathfrak{s}(\varepsilon)$. If

$$\sup\{\|s_\varepsilon\|_{L^2} | \; \varepsilon \in (\varepsilon_\mathfrak{s}, 1]\} < \infty \; ,$$

then the strong limit $\lim_{\varepsilon \to \varepsilon_\mathfrak{s}} s_\varepsilon$ exists in L_2^p, and is a smooth solution of the equation $l(\varepsilon_\mathfrak{s}, \cdot) = 0$.

2. For any maximal element $\mathfrak{s} \in \mathcal{S}$, one has $\varepsilon_\mathfrak{s} = 0$.

Proof:
1. By assumption and the third statement in Lemma 4.5 one gets an uniform bound

$$m_{s_\varepsilon} = \sup_X |s_\varepsilon| \leq M \; , \; \forall \, \varepsilon \in (\varepsilon_\mathfrak{s}, 1] \; . \tag{55}$$

Therefore, the family $(s_\varepsilon)_{\varepsilon \in (\varepsilon_\mathfrak{s}, 1]}$ is bounded in L_2^p by Proposition 4.12. Using now the inequality (53), we see that $\frac{d}{d\varepsilon} s_\varepsilon$ is also bounded in L_2^p, hence the map $\mathfrak{s} : (\varepsilon_\mathfrak{s}, 1] \to L_2^p(\hat{P} \times_{\mathrm{ad}} i\mathfrak{k}_\beta^\perp)$ is Lipschitz. Therefore one has a strong limit

$$s = \lim_{\varepsilon \to \varepsilon_\mathfrak{s}} s_\varepsilon \; ,$$

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which will be a solution of the equation $l(\varepsilon_{\mathfrak{s}}, \cdot) = 0$ by Remark 4.7, and will be smooth by Lemma 4.6.

2. If $\varepsilon_{\mathfrak{s}} > 0$, the second statement of Lemma 4.5 gives an uniform bound of the form (55). Therefore, the first part of this proposition applies and gives a strong L_2^p - limit $s = \lim_{\varepsilon \to \varepsilon_{\mathfrak{s}}} s_\varepsilon$ which is smooth and solves the equation $l(\varepsilon_{\mathfrak{s}}, \cdot) = 0$.

Using Lemma 4.8 and the implicit function theorem we get a smooth extension of $\mathfrak{s}$ on a larger interval, contradicting maximality. ∎

This gives immediately the following crucial result:

Theorem 4.14
1. There exists a smooth family $(s_\varepsilon)_{\varepsilon \in (0,1]}$ with $l(\varepsilon, s_\varepsilon) \equiv 0$.
2. If $\sup\{\|s_\varepsilon\|_{L^2} \mid \varepsilon \in (0,1]\} < \infty$, then s_ε converges strongly in L_2^p to a smooth solution s of the equation (25) as $\varepsilon \to 0$.

Therefore, when $\sup\{\|s_\varepsilon\|_{L^2} \mid \varepsilon \in (0,1]\} < \infty$, one gets a Hermitian-Einstein reduction $\hat{P}_s := e^{-\frac{s}{2}}(\hat{P}) \in \mathcal{R}_{P_0}(\hat{Q})$.

4.4 Destabilizing the pair in the unbounded case

4.4.1 Estimates in the unbounded case

Let $(s_\varepsilon)_{\varepsilon \in (0,1]}$ be the smooth family given by Theorem 4.14. Our first aim is to get a uniform L_1^2 - bound for the L^2-normed section

$$\sigma_\varepsilon := \frac{s_\varepsilon}{\| \, s_\varepsilon \, \|_{L^2}} \quad .$$

By the third statement of Lemma 4.5, we get uniform bounds

$$\sup_X |\sigma_\varepsilon| \leq M \ , \ \sup_{x \in X} \{|\lambda| \mid \lambda \in \mathrm{Spec}([(\sigma_\varepsilon)_x, \cdot])\} \leq M \tag{56}$$

for all $\varepsilon > 0$ for which $\|s_\varepsilon\|_{L^2} \geq 1$.

Remark 4.15 *Fix $\alpha \in (0,1)$. Then, for any $u \in \mathbb{R}$, $n > 0$ one has*

1.
$$|e^{nu} - 1| \geq \begin{cases} \alpha n |u| & \text{if} \quad nu \geq \log(\alpha) \\ (1 - \alpha) & \text{if} \quad nu \leq \log(\alpha) \ . \end{cases}$$

2. *Supposing $|u| \leq M$, we get*

$$|e^{nu} - 1| \geq \left[\min\left(n\alpha, \frac{1-\alpha}{M} \right) \right] |u| \ , \ \ \forall u \in [-M, M] \ .$$

3. *Choosing $\alpha = \frac{1}{2}$, we get*

(a)

$$|e^{nu} - 1| \geq \frac{1}{2M} |u| \ , \ \ \forall n \geq \frac{1}{M} \ , \ \ \forall \, u \in [-M, M] \ .$$

(b)

$$\frac{1}{2M} \leq n\eta^2(nu) \ , \ \ \ \forall n \geq \frac{1}{M} \ , \ \ \forall \, u \in [-M, M] \ .$$

(c) *Let V be Hermitian space. Then*

$$\frac{1}{2M} \|x\|^2 \leq n \|\eta(nh)(x)\|^2$$

for all $n \geq \frac{1}{M}$ and for every Hermitian endomorphism $h \in \mathrm{Herm}(V)$ with the property $\mathrm{Spec}(h) \subset [-M, M]$.

Proof: The first statements are straightforward. The last one follows from the previous ones and Remark 6.15 in the following way: we take

$$f := \sqrt{\frac{1}{2M}} \ \mathrm{id}_V \ , \ \ g(t) := \sqrt{n} \ \eta(nt) \ ,$$

and we notice that $|f|_{[-M,M]} \leq |g|_{[-M,M]}$ by 3b. ∎

We divide (34) by $\|s\|_{L^2}$, taking into account Remark 4.15 3c and (56). We get, for $\|s\|_{L^2} \geq \max(\frac{1}{M}, 1)$,

$$\frac{1}{2M} |\partial_0 \sigma|^2 \leq \|s\| |\eta(\|s\|[\sigma, \cdot])(\partial_0 \sigma)|^2 = \frac{1}{\|s\|} |\eta([s, \cdot](\partial_0 s)|^2 \leq$$

$$\leq \frac{1}{\|s\|} \left[-\frac{1}{2} P(|s|^2) - (k^0, s) \right] \ .$$

Integrating on X the obtained inequality, we get

$$\|\partial_0 \sigma\|_{L^2}^2 \leq 2M \|k^0\|_{L^2} \ . \tag{57}$$

Concluding, we can state

Proposition 4.16 *Suppose that* $(s_\varepsilon)_{\varepsilon \in (0,1]}$ *is not L^2-bounded. Then the set*

$$\left\{ \sigma_\varepsilon \mid \|s_\varepsilon\|_{L^2} \geq \max\left(\frac{1}{M}, 1 \right) \right\}$$

is uniformly bounded in $L^\infty \cap L_1^2$. In particular, there exists a sequence $(\varepsilon_n)_n$, $\varepsilon_n \to 0$ such that

1. *$\lim_{n\to\infty} \|s_{\varepsilon_n}\|_{L^2} = \infty$.*

2. *The sequence $(\sigma_{\varepsilon_n})_n$ converges*

   - *weakly in L_1^2 ,*
   - *strongly in L^2 ,*
   - *almost everywhere on X*

   to an $L_1^2 \cap L^\infty$ - section σ in the bundle $\hat{P}^0 \times_{\mathrm{ad}} i\mathfrak{k}_\beta^\perp$.

4.4.2 The properties of the limit σ

The aim of this paragraph is the following

Theorem 4.17

1. *$\|\sigma\|_{L^2} = 1$.*

2. *The map $X \ni x \mapsto [\sigma(x)] \in i\mathfrak{k}/\mathrm{ad}_{\hat{K}}$ which assigns to every $x \in X$ the conjugacy class of $\sigma(x)$, is constant almost everywhere.*

3. *$\mathrm{pr}_{V^+_{\sigma(x)}}(\bar\partial \sigma) = 0$ for almost every $x \in X$, where $V^+_{\sigma(x)}$ is the direct sum of the eigenspaces corresponding to strictly positive eigenvalues of the endomorphism $[\sigma(x), \cdot]$ on $\mathfrak{g}_x := \hat{P}_x \times_{\mathrm{ad}} \mathfrak{g}$.*

In order to prove this, we will need several preparations:

Let $s \in A^0(\hat{P} \times_{\mathrm{ad}} i\mathfrak{k})$ be a smooth section. Denote by V_s^0 the linear fibration $\bigcup_{x \in X} V_{s,x}^0$, where $V_{s,x}^0 = \ker[s_x, \cdot] \subset \hat{P}_x \times_{\mathrm{ad}} \mathfrak{g}$.

Let X_k^s be the closed subset of X where $\ker[s_x, \cdot]$ has rank $\leq k$. Then $\ker([s, \cdot])$ defines a continuous subbundle of $\hat{P} \times_{\mathrm{ad}} \mathfrak{g}|_{X_k^s \setminus X_{k-1}^s}$. Therefore, the projection $p_{V_s^0}$ is continuous on every $X_k^s \setminus X_{k-1}^s$, hence it is a bounded measurable section.

With these remarks, we can state

Proposition 4.18 *Let $\hat{A}^0 \in \mathcal{A}(\hat{P})$ and let $(\sigma_n)_n$, $\sigma_n \in A^0(\hat{P} \times_{\mathrm{ad}} i\mathfrak{k})$ be a sequence of smooth sections with the following properties:*

1. *It is weakly convergent in L_1^2.*

2. *It is bounded in L^∞.*

3. *It converges strongly in L^2.*

4. *It converges almost everywhere on X.*

5. *$(\|\mathrm{pr}_{V_{\sigma_n}^0}[\partial_0\sigma_n]\|_{L^2})_n$ converges to 0.*

Then the weak limit σ of this sequence (which belongs to $L_1^2 \cap L^\infty$) defines an almost everywhere constant map $x \mapsto [\sigma(x)] \in \mathfrak{k}/\hat{K}$.

Proof: Let Let $\iota_1, \ldots, \iota_k \in \mathbb{C}[\hat{\mathfrak{g}}]$ be homogeneous generators of the invariant algebra $\mathbb{C}[\hat{\mathfrak{g}}]^{\hat{G}}$. We will prove that $\iota_i(\sigma)$ are constant almost everywhere, and the claim will follow from Corollary 6.8 in the Appendix (compare with the proof of 6.9).

Let $j_i : \hat{\mathfrak{g}}^{d_i} \to \mathbb{C}$ be the $\mathrm{ad}_{\hat{G}}$-invariant symmetric multilinear map which corresponds to ι_i

Using properties 2. and 3. and 4. one gets easily, by the Lebesgue dominant convergence theorem, that $\iota_i(\sigma_n) = j_i(\sigma_n, \ldots, \sigma_n)$ converges to $\iota_i(\sigma)$ strongly in L^2.

One has
$$\partial[j_i(\sigma_n, \ldots, \sigma_n)] = d_i j_i(\partial_0\sigma_n, \sigma_n, \ldots, \sigma_n) \ .$$

On the other hand, by 2. and 3. the sequence of sections
$$j_i(\cdot, \sigma_n, \ldots, \sigma_n) \in A^0(\hat{P} \times_{\mathrm{ad}} \mathrm{Hom}(i\mathfrak{k}, \mathbb{C}))$$

is L^∞ - bounded and converges strongly in L^2 to $j_i(\cdot, \sigma, \ldots, \sigma)$. Taking into account 1., 2. and 3. we deduce by Lemma 6.17 in Appendix that

$$d_i j_i(\partial_0\sigma_n, \sigma_n, \ldots, \sigma_n) \to d_i j_i(\partial_0\sigma, \sigma, \ldots, \sigma) \text{ weakly in } L^2 \ .$$

On the other hand $d_i j_i(\partial_0\sigma, \sigma, \ldots, \sigma) = \partial[j_i(\sigma, \ldots, \sigma)]$ as distributions because Leibniz's rule extends to $L_1^2 \cap L^\infty$-sections.[3]

Therefore,

$$d_i j_i(\partial_0\sigma_n, \sigma_n, \ldots, \sigma_n) \to \partial[j_i(\sigma, \ldots, \sigma)] \ \text{ weakly in } L^2 \ . \tag{58}$$

On the other hand, taking into account that j_i is $\mathrm{ad}_{\hat{G}}$-invariant (hence in particular ad_G-invariant), it follows easily that, for any $v \in \mathfrak{g}$, the linear functional

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[3]This can be proved by constructing a sequence of smooth sections converging to σ with respect to both L_1^2 - and L^p - norms, for p sufficiently large.

$j_i(\cdot, v, \ldots, v)$ vanishes on $[v, \mathfrak{g}]$. The point is that, when $v \in i\mathfrak{k}$, one has an orthogonal decomposition

$$\mathfrak{g} = [v, \mathfrak{g}] \oplus z_{\mathfrak{g}}(v) \ .$$

This follows from the fact that, in this case, $z_{\mathfrak{g}}(v)$ is just the complexification of $z_{\mathfrak{k}}(iv)$.

Therefore one gets pointwise

$$d_i j_i(\partial_0 \sigma_n, \sigma_n, \ldots, \sigma_n) = d_i j_i(\mathrm{pr}_{V_{\sigma_n}^0}(\partial_0 \sigma_n), \sigma_n, \ldots, \sigma_n) \ ,$$

and the right term converges to 0 in L^2 as $n \to \infty$ by 5. and 2. By (58) we get that $\partial[j_i(\sigma, \ldots, \sigma)] = 0$ in the week sense, hence $j_i(\sigma, \ldots, \sigma)$ is a constant. The claim follows now from Corollary 6.8 ∎

For an element $u \in i\mathfrak{k}$ we denote by V_u^- the direct sum of eigenspaces of $[u, \cdot] \in \mathrm{Herm}(\mathfrak{g})$ corresponding to strictly negative eigenvalues. Again, for a section $u \in A^0(\hat{P} \times_{\mathrm{ad}} i\mathfrak{k})$, the projection $\mathrm{pr}_{V_u^-}$ on the fibration defined by u is in $L^\infty(\mathrm{Herm}(\hat{P} \times_{\mathrm{ad}} \mathfrak{g}))$.

Proposition 4.19 *Let $\hat{A}^0 \in \mathcal{A}(\hat{P})$ and let $(\sigma_n)_n$, $\sigma_n \in A^0(\hat{P} \times_{\mathrm{ad}} i\mathfrak{k})$ be a sequence of smooth sections with the following properties*

1. *It is weakly convergent in L_1^2.*

2. *It is bounded in L^∞.*

3. *It converges strongly in L^2.*

4. *It converges almost everywhere on X.*

5. *The weak limit σ of this sequence defines an (almost everywhere) constant conjugacy class $[\sigma] \in i\mathfrak{k}/\hat{K}$.*

6. *$(\|\mathrm{pr}_{V_{\sigma_n}^-}[\partial_0 \sigma_n]\|_{L^2})_n$ converges to 0.*

Then the weak limit σ (which belongs to $L_1^2 \cap L^\infty$) satisfies

$$\mathrm{pr}_{V_{\sigma,x}^-}[\partial_0 \sigma] = 0 \ \ for \ almost \ every \ x \in X \ .$$

Proof: Let $\sigma_0 \in i\mathfrak{k}$ be a representative of $[\sigma]$ and $\lambda_1 < \ldots \lambda_k < 0$ be the negative eigenvalues of $[\sigma_0, \cdot]$. Then $[\sigma_x, \cdot]$ has the same eigenvalues with the same multiplicities for almost every $x \in X$.

Indeed, one can easily see that for $u \in i\mathfrak{k}$, the eigenvalues of the endomorphism $[u, \cdot] \in \mathrm{Herm}(\mathfrak{g})$ depend only on the class $[u] \in i\mathfrak{k}/\hat{K}$. This follows from the fact that, for $\hat{k} \in \hat{K}$, one has $[\mathrm{ad}_{\hat{k}}(s), \cdot] = \mathrm{ad}_{\hat{k}}[s, \mathrm{ad}_{\hat{k}^{-1}}(\cdot)]$.

Let $\eta \in (\lambda_k, 0)$ and $\chi : \mathbb{R} \to [0,1]$ a smooth increasing function which is 1 on $(-\infty, \eta]$ and 0 on $[0, \infty)$.

Put
$$q_n := \chi([\sigma_n, \cdot]) .$$

Since σ_n converges strongly in L^2, it follows that q_n converges strongly in L^2 to $q := \chi([\sigma, \cdot]) = \mathrm{pr}_{V_\sigma^-}$. This follows from the Lebesgue dominated convergence theorem, from 4. and the fact that, for any Hermitian vector space E, the map

$$\chi : \mathrm{Herm}(E) \longrightarrow \mathrm{Herm}(E) , \ f \mapsto \chi(f)$$

is continuous.

Therefore $q_n[\partial_0 \sigma_n]$ converges to $q(\partial_0(\sigma)) = \mathrm{pr}_{V_\sigma^-}[\partial_0 \sigma]$ as distributions. On the other hand
$$\| q_n[\partial_0 \sigma_n] \|_{L^2} \leq \|\mathrm{pr}_{V_{\sigma_n}^-}[\partial_0 \sigma_n]\|_{L^2} ,$$

which converges to 0 by 6.

Therefore $\mathrm{pr}_{V_\sigma^-}[\partial_0 \sigma] = 0$ as a distribution, hence as an L^2 - section as well. ∎

We can now give the

Proof (of Theorem 4.17):

We make use of Proposition 4.16 and put $s_n := s_{\varepsilon_n}$. $\sigma_n := \sigma_{\varepsilon_n}$. Since $\|\sigma_n\|_{L^2} = 1$, and $\sigma_n \to \sigma$ strongly in L^2, we get $\|\sigma\|_{L^2} = 1$ as claimed.

For the proof of 2. and 3. we claim first that

$$(\|\mathrm{pr}_{V_-^{\sigma_n}}[\partial_0 \sigma_n]\|_{L^2})_n \to 0 , \tag{59}$$

where , for $s \in A^0(\hat{P} \times_{\mathrm{ad}} i\mathfrak{k})$, we denoted by V_-^s the fibration whose fibre in $x \in X$ is the direct sum of all eigenspaces corresponding to all non-positive eigenvalues of $[s_x, \cdot]$.

This follows easily in the following way: using the inequality

$$\frac{1 - e^{-t}}{t} \geq 1 \text{ for } t \leq 0 ,$$

we see that $\eta \geq \chi_-$, where $\chi_- := \chi_{\mathbb{R}_{\leq 0}}$ is the characteristic function of the set $\mathbb{R}_{\leq 0}$. By Remark 6.15 and (34) one gets

$$\left| \mathrm{pr}_{V_-^{\sigma_n}}[\partial_0 \sigma_n] \right|_x^2 = \frac{1}{\|s_n\|^2} \left| \mathrm{pr}_{V_-^{s_n}}[\partial_0 s_n] \right|_x^2 = \frac{1}{\|s_n\|^2} |(\chi_-([s_n, \cdot])(\partial_0 s_n)|_x^2 \leq$$

$$\leq \frac{1}{\|s_n\|^2} |(\eta([s_n, \cdot])(\partial_0 s_n)|_x^2 \leq \frac{1}{\|s_n\|^2} \left[-\frac{1}{2}P(|s_n|^2) - \varepsilon|s_n|^2 - (k^0(x), s_n) \right] (x) .$$

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It suffices to integrate the obtained inequality over X, and let $\varepsilon \to 0$ taking into account that $\|s_n\| \to \infty$.

In particular, it follows that

$$(\|\mathrm{pr}_{V_{\sigma_n}^0}[\partial_0 \sigma_n]\|_{L^2})_n \to 0 \ ,$$

hence, by Propositions 4.18, the limit σ defines an almost everywhere constant map $X \to i\mathfrak{k}/\hat{K}$. Therefore, we can apply Proposition 4.19, because (59) obviously implies $\|\mathrm{pr}_{V_{\sigma_n}^-}[\partial_0 \sigma_n]\|_{L^2} \to 0$. ∎

4.4.3 The limit σ destabilizes

Let σ be the limit given by Proposition 4.16.

Proposition 4.20 *Let $[\sigma] \subset i\mathfrak{k}$ be the element in $i\mathfrak{k}/\hat{K}$ defined by the (almost everywhere constant) map $x \mapsto [\sigma_x]$ (modulo $\mathrm{ad}_{\hat{K}}$) and let $\sigma_0 \in i\mathfrak{k}_\beta^\perp \setminus \{0\}$ be any representative of $[\sigma]$. Then:*

1. *The $L_1^2 \cap L^\infty$ section σ is smooth and takes values in $[\sigma]$ on the complement X_ρ of a Zariski subset of codimension at least 2 such that the subspace*

$$\{q \in \hat{\mathcal{Q}}|_{X_\rho}, \ \sigma(q) \in \hat{\mathfrak{g}}_{\sigma_0}\}$$

   extends to a meromorphic reduction $\hat{\mathcal{Q}}^\rho \hookrightarrow \hat{\mathcal{Q}}$ of $\hat{\mathcal{Q}}$ to $\hat{G}_{\sigma_0}$ which is holomorphic over X_ρ.

2. *The section $s(\sigma_0, \rho)$ associated with σ_0 and the $Z_{\hat{K}}(\sigma_0)$-reduction $\hat{\mathcal{Q}}^\rho \cap \hat{P}$ of $\hat{P}|_{X_\rho}$ coincides with σ. More precisely, $\sigma(q) = \sigma_0$ for all $q \in \hat{\mathcal{Q}}^\rho \cap \hat{P}$.*

Proof: This follows from Proposition 4.17 and Proposition 6.27. There is one simple detail to be explained. The weak holomorphy condition in Proposition 6.27 reads in our case

$$\mathrm{pr}_{V_{\hat{\sigma}(x)}^+} (\bar{\partial}_x \hat{\sigma}) = 0 \text{ for almost every } x \in X$$

where $\hat{\sigma}$ is the section in $\hat{P} \times_{\mathrm{ad}} i\hat{\mathfrak{g}} = i\mathrm{ad}(\hat{P})$ defined by σ, and $V_{\hat{\sigma}(x)}^+$ is the direct sum of strictly positive eigenspaces of $[\hat{\sigma}(x), \cdot] \in \mathrm{Herm}(\hat{\mathfrak{g}}_x)$.

But, since
$$\bar{\partial}\hat{\sigma} \in L^2(\Lambda^{01}(\hat{P} \times_{\mathrm{ad}} \mathfrak{g})) \ , \ V_{\hat{\sigma}(x)}^+ = V_{\sigma(x)}^+ \subset \mathfrak{g}_x \ ,$$

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this condition is equivalent to

$$\mathrm{pr}_{V_{\sigma(x)}^+}(\bar{\partial}_x\sigma) = 0 \text{ for almost every } x \in X .$$

∎

The following result will end the proof of Theorem 4.2.

Theorem 4.21 *The pair* $(\rho, \frac{\sigma_0}{2})$ *destabilizes the pair* $(\hat{J}, \varphi)$.

The first step is to give an explicit formula for the total maximal weight

$$\frac{2\pi}{(n-1)!}\deg_g\left(\rho, h(\frac{\sigma_0}{2})\right) + \int_{X_0} \lambda^{\frac{\sigma_0}{2}}(\varphi, \rho)vol_g .$$

For the first term, we obtain by (14)

$$\frac{2\pi}{(n-1)!}\deg_g\left(\rho, h(\frac{\sigma_0}{2})\right) = \frac{1}{2(n-1)!}\int_X h\left(\left[iF_{\hat{A}^0} - \frac{i}{2}[a \wedge a]\right], \sigma\right) \wedge \omega_g^{n-1} =$$

$$\frac{1}{2}\int_X h\left(\Lambda_g\left[iF_{\hat{A}^0} - \frac{i}{2}[a \wedge a]\right], \sigma\right) \wedge vol_g ,$$

where a is the second fundamental form of the $\hat{K}_{\sigma_0} = \hat{G}(\sigma_0) \cap \hat{K}$-bundle $\hat{\mathcal{Q}}^\rho \cap \hat{P}$ in $\hat{P}|_{X_\rho}$.

By (19) we have

$$\mathrm{pr}_{\hat{\mathfrak{k}}_{\sigma_0}}[a \wedge a] = -2 \sum_{\lambda \in \mathrm{Spec}^+[\sigma_0, \cdot]} [a_\lambda \wedge \bar{a}_\lambda] ,$$

so using the known formula (21), we get

$$h(\frac{i}{2}\Lambda_g\mathrm{pr}_{\hat{\mathfrak{k}}_{\sigma_0}}[a \wedge a], \sigma) = - \sum_{\lambda \in \mathrm{Spec}^+[\sigma_0, \cdot]} i\Lambda_g h([a_\lambda \wedge \bar{a}_\lambda], \sigma) =$$

$$= - \sum_{\lambda \in \mathrm{Spec}^+[\sigma_0, \cdot]} i\lambda\Lambda_g h(a_\lambda \wedge \bar{a}_\lambda) = - \sum_{\lambda \in \mathrm{Spec}^+[\sigma_0, \cdot]} \lambda|a_\lambda|^2 . \qquad (60)$$

The (1,0)-form $\partial_0\sigma$ has non-trivial components only on the eigenspaces of $[\sigma_0, \cdot]$ associated with strictly positive eigenvalues. Indeed, since the conjugacy class of σ is constant, $\partial_0\sigma$ has no $\ker[\sigma_0, \cdot]$ - component by Proposition 6.9. On the other hand, by the holomorphy of $\hat{\mathcal{Q}}^\rho|_{X_\rho}$ it follows that $a_\lambda^{10} = 0$ for $\lambda \in \mathrm{Spec}^-[\sigma_0, \cdot]$ (see formula (18), Proposition 6.26).

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Therefore
$$\partial_0 \sigma = \sum_{\lambda \in \mathrm{Spec}^+[\sigma_0,\cdot]} [a_\lambda, \sigma] = - \sum_{\lambda \in \mathrm{Spec}^+[\sigma_0,\cdot]} \lambda a_\lambda . \tag{61}$$

Let ϕ be the real function defined by
$$\phi(\tau) = \begin{cases} \frac{1}{\sqrt{\tau}} & \text{if} \quad \tau > 0 \\ 0 & \text{if} \quad \tau \le 0 . \end{cases}$$

We get
$$\phi([\sigma,\cdot])(\partial_0 \sigma) = - \sum_{\lambda \in \mathrm{Spec}^+[\sigma_0,\cdot]} \sqrt{\lambda}\, a_\lambda ,$$

so, comparing with (60), one can write
$$h(\frac{i}{2} \Lambda_g \mathrm{pr}_{\hat{\mathfrak{k}}_{\sigma_0}} [a \wedge a], \sigma) = - \left| \phi([\sigma_0,\cdot])(\partial_0 \sigma) \right|^2 .$$

Since the (1,0)-form $\partial_0 \sigma$ has non-trivial components only on the eigenspaces of $[\sigma, \cdot]$ associated with strictly positive eigenvalues, the right hand term is indeed smooth on X_ρ.

Therefore, we obtain for the total maximal weight of the pair $(\hat{J}, \varphi)$ the formula
$$\lambda((\hat{J}, \varphi), (\rho, \frac{\sigma_0}{2})) := \frac{2\pi}{(n-1)!} \deg_g(\rho, h(\frac{\sigma_0}{2})) + \int_{X_\rho} \lambda^{\frac{\sigma_0}{2}}(\varphi, \rho) vol_g =$$
$$= \frac{1}{2} \left[\langle i \Lambda_g F_{\hat{A}^0}, \sigma \rangle_{L^2} + \| \phi([\sigma,\cdot])(\partial_0 \sigma) \|_{L^2}^2 \right] + \int_{X_\rho} \lim_{t \to \infty} \mu^{\frac{-i\sigma}{2}}(e^{t\frac{\sigma}{2}} \varphi) .$$

Lemma 4.22 *Let (ρ, σ_0) be any pair consisting of an element $\sigma_0 \in i\mathfrak{k}$ and a meromorphic reduction of $\hat{Q}$ to $\hat{G}(\sigma_0)$. Fix a $\hat{K}$-reduction $\hat{P}$ of $\hat{Q}$, let σ be the section in $A^0(\hat{P} \times_{\mathrm{ad}} i\mathfrak{k})$ which corresponds to σ_0 and this reduction. Let $\hat{A}^0 \in \mathcal{A}(\hat{P})$ be the Chern connection of $\hat{P}$ and $d_{\hat{A}^0} = \bar{\partial} + \partial_0$ the associated covariant derivative. Then*
$$\lambda((\hat{J}, \varphi), (\rho, \frac{\sigma_0}{2})) =$$
$$\frac{1}{2} \langle i \Lambda_g F_{\hat{A}^0}, \sigma \rangle_{L^2} + \lim_{t \to \infty} \int_{X_\rho} \left[\frac{1}{2} t |\eta([t\sigma,\cdot])(\partial_0 \sigma)|^2 + \mu^{-i\frac{\sigma}{2}}(e^{t\frac{\sigma}{2}} \varphi) \right] vol_g .$$

Proof: It suffices to prove that
$$\| \phi([\sigma,\cdot])(\partial_0(\sigma)) \|_{L^2}^2 = \lim_{t \to \infty} \int_{X_\rho} t\, |\eta([t\sigma,\cdot])(\partial_0 \sigma)|^2\, vol_g .$$

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For $t > 0$, consider the real analytic function η_t defined by

$$\eta_t(\tau) = \sqrt{t}\, \eta(t\tau) = \begin{cases} \sqrt{\dfrac{1-e^{-t\tau}}{\tau}} & \text{if} \quad \tau \neq 0 \\ \sqrt{t} & \text{if} \quad \tau = 0 \; . \end{cases} \tag{62}$$

Recall that the conjugacy class of σ is constant, so the eigenvalues of $[\sigma, \cdot]$ are also constant (see the proof of Proposition 4.19). Let λ_0 be the first positive eigenvalue of $[\sigma, \cdot]$. Note that, by Proposition 6.9 and Theorem 6.26, the form $\partial_0(\sigma)|_{X_\rho}$ has non-trivial components only along the <u>strictly</u> positive eigenspaces of $[\sigma, \cdot]$. Therefore, pointwise on X_ρ one has

$$\left| t\, |\eta([t\sigma, \cdot])(\partial_0\sigma)|^2 - |\phi([\sigma, \cdot])(\partial_0\sigma)|^2 \right| \leq \sup_{[\lambda_0, \infty]} |\eta_t^2 - \phi^2| |\partial_0\sigma|^2 \; .$$

The point is now that η_t^2 converges uniformly to ϕ^2 on the interval $[\lambda_0, \infty)$ as $t \to \infty$. Integrating both sides on X_ρ we get the result. $\blacksquare$

Let $\sigma_n = \frac{1}{\|s_n\|_{L^2}} s_n$ be the sequence with limit σ given by Proposition 4.16.

Lemma 4.23 *Fix $t_0 > 0$. Then*

$$\langle i\Lambda_g F_{\hat{A}^0}, \sigma \rangle_{L^2} + \int_{X_\rho} \left[\frac{1}{2} t_0 |\eta([t_0\sigma, \cdot])(\partial_0\sigma)|^2 + \mu^{-i\frac{\sigma}{2}}(e^{t_0\frac{\sigma}{2}}\varphi) \right] vol_g \leq$$

$$\liminf_{n \to \infty} \left\{ \langle i\Lambda_g F_{\hat{A}^0}, \sigma_n \rangle_{L^2} + \int_{X_\rho} \left[\frac{1}{2} t_0 |\eta([t_0\sigma_n, \cdot])(\partial_0\sigma_n)|^2 + \mu^{-i\frac{\sigma_n}{2}}(e^{t_0\frac{\sigma_n}{2}}\varphi) \right] vol_g \right\} \; .$$

Proof: First of all notice that

$$\langle i\Lambda_g F_{\hat{A}^0}, \sigma_n \rangle_{L^2} \to \langle i\Lambda_g F_{\hat{A}^0}, \sigma \rangle_{L^2} \; , \tag{63}$$

because $(\sigma_n)_n$ converges strongly in L^2 to σ.

Second, one has

$$\mu^{-i\frac{\sigma}{2}}(e^{t_0\frac{\sigma}{2}}\varphi) = \langle \mu(\varphi), -i\frac{\sigma}{2} \rangle + [\mu^{-i\frac{\sigma}{2}}(e^{t_0\frac{\sigma}{2}}\varphi) - \mu^{-i\frac{\sigma}{2}}(\varphi)] \; ,$$

$$\mu^{-i\frac{\sigma_n}{2}}(e^{t_0\frac{\sigma_n}{2}}\varphi) = \langle \mu(\varphi), -i\frac{\sigma_n}{2} \rangle + [\mu^{-i\frac{\sigma_n}{2}}(e^{t_0\frac{\sigma_n}{2}}\varphi) - \mu^{-i\frac{\sigma_n}{2}}(\varphi)] \; .$$

Since $\sigma_n \to \sigma$ strongly in L^2, it follows that

$$\int_{X_\rho} \langle \mu(\varphi), -i\frac{\sigma_n}{2} \rangle vol_g \to \int_{X_\rho} \langle \mu(\varphi), -i\frac{\sigma}{2} \rangle vol_g \; . \tag{64}$$

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The two terms $[\mu^{-i\frac{\sigma_n}{2}}(e^{t_0\frac{\sigma_n}{2}}\varphi)-\mu^{-i\frac{\sigma_n}{2}}(\varphi)]$, $[\mu^{-i\frac{\sigma}{2}}(e^{t_0\frac{\sigma}{2}}\varphi)-\mu^{-i\frac{\sigma}{2}}(\varphi)]$ are *pointwise* non-negative, because the value in every point $x \in X$ can be identified with the energy of a curve in the fibre F_x. On the other hand, the first term converges almost everywhere to the second as $n \to \infty$. By Fatou's lemma, it follows that

$$\int_{X_\rho} [\mu^{-i\frac{\sigma}{2}}(e^{t_0\frac{\sigma}{2}}\varphi) - \mu^{-i\frac{\sigma}{2}}(\varphi)]vol_g \leq$$

$$\leq \liminf_{n\to\infty} \int_{X_\rho} [\mu^{-i\frac{\sigma_n}{2}}(e^{t_0\frac{\sigma_n}{2}}\varphi) - \mu^{-i\frac{\sigma_n}{2}}(\varphi)] \ . \tag{65}$$

Finally we state that

$$\int_{X_\rho} \left[|\eta([t_0\sigma,\cdot])(\partial_0\sigma)|^2\right] vol_g \leq \liminf_{t\to\infty} \int_{X_\rho} \left[|\eta([t_0\sigma_n,\cdot])(\partial_0\sigma_n)|^2\right] vol_g \ . \tag{66}$$

Indeed, since $(\eta([t_0\sigma_n,\cdot]))_n$ converges strongly in L^2 to $\eta([t_0\sigma_n,\cdot])$ and is L^∞-bounded, and since $\partial_0(\sigma)$ converges weakly in L^2 to $\partial_0\sigma$, it follows that $\eta([t_0\sigma_n,\cdot])(\partial_0\sigma_n)$ converges weakly to $\eta([t_0\sigma,\cdot])(\partial_0\sigma)$ in L^2 (see Lemma 6.17 in the Appendix).

Now the inequality (66) follows from the well known semicontinuity property of the norm with respect to the weak convergence in a Hilbert space. The claim of our lemma follows from (63), (64), (65) and (66). ∎

Lemma 4.24 *Let* $\mathfrak{s} \in A^0(\hat{P} \times_{\mathrm{ad}} i\mathfrak{k})$. *Then the maps*

$$t_0 \mapsto t_0|\eta([t_0\mathfrak{s},\cdot])(\partial_0\mathfrak{s})|^2 \ ,$$

$$t_0 \mapsto \mu_0^{-i\frac{\mathfrak{s}}{2}}(e^{t_0\frac{\mathfrak{s}}{2}}\varphi)$$

are pointwise increasing.

Proof: The second map is obviously pointwise increasing (see Remark 1.6). For the first, it suffices to notice that the function η_{t_0} defined by (62) takes positive values and is increasing with respect to t_0. The result follows from Remark 6.15. We can now give the

Proof (of Theorem 4.21):

If not, one would have

$$\lambda((\hat{J},\varphi),(\rho,\frac{\sigma_0}{2})) > 0 \ .$$

By Lemma 4.22, it would follow that there exists $t_0 > 0$ such that

$$U := \frac{1}{2}\langle i\Lambda_g F_{\hat{A}^0}, \sigma\rangle_{L^2} + \int_{X_\rho} \left[\frac{1}{2}t_0|\eta([t_0\sigma, \cdot])(\partial_0\sigma)|^2 + \mu^{-i\frac{\sigma}{2}}(e^{t_0\frac{\sigma}{2}}\varphi)\right] vol_g > 0 \ .$$

By Lemma 4.23 we get that for all sufficiently large $n \in \mathbb{N}$, we have

$$\frac{1}{2}\langle i\Lambda_g F_{\hat{A}^0}, \sigma_n\rangle_{L^2} + \int_X \left[\frac{1}{2}t_0|\eta([t_0\sigma_n, \cdot])(\partial_0\sigma_n)|^2 + \mu^{-i\frac{\sigma_n}{2}}(e^{t_0\frac{\sigma_n}{2}}\varphi)\right] vol_g > \frac{U}{2} \ .$$

Choose n sufficiently large such that $\|s_n\|_{L^2} > t_0$. By the monotony Lemma 4.24, we get that

$$\frac{1}{2}\langle i\Lambda_g F_{\hat{A}^0}, \sigma_n\rangle_{L^2} +$$

$$+ \int_X \left[\frac{1}{2}\|s_n\||\eta([\|s_n\|\sigma_n, \cdot])(\partial_0\sigma_n)|^2 + \mu^{-i\frac{\sigma_n}{2}}(e^{\|s_n\|\frac{\sigma_n}{2}}\varphi)\right] vol_g > \frac{U}{2} \ ,$$

or, equivalently,

$$\frac{1}{\|s_n\|}\left\{\frac{1}{2}\langle i\Lambda_g F_{\hat{A}^0}, s_n\rangle_{L^2} + \int_X \left[\frac{1}{2}|\eta([s_n, \cdot])(\partial_0 s_n)|^2 + \mu^{-i\frac{s_n}{2}}(e^{\frac{s_n}{2}}\varphi)\right] vol_g\right\} > \frac{U}{2}.$$

But, integrating (32) on X and taking into account the identity given by Proposition 6.16, we get

$$\frac{1}{\|s_n\|}\left\{\frac{1}{2}\langle i\Lambda_g F_{\hat{A}^0}, s_n\rangle_{L^2} + \int_X \left[\frac{1}{2}|\eta([s_n, \cdot])(\partial_0 s_n)|^2 + \mu^{-i\frac{s_n}{2}}(e^{\frac{s_n}{2}}\varphi)\right] vol_g\right\} =$$

$$= -\frac{\varepsilon\|s_n\|}{2} \ .$$

∎

We now can finally give the proof of our main result.

Proof (of Theorem 4.2): Theorem 4.14 yields a Hermitian-Einstein reduction $\hat{P}_s$, unless the smooth family $(s_\varepsilon)_{\varepsilon \in (0,1]}$ given by this theorem is not L^2-bounded. In this case Proposition 4.16 applies and one obtains a pair $(\rho, \frac{\sigma_0}{2})$ with

$$\lambda((\hat{J}, \varphi), (\rho, \frac{\sigma_0}{2})) \leq 0$$

by Theorem 4.21. But, using Remark 2.20 and the fact that $\sigma_0 \in i\mathfrak{k}_\beta^\perp \setminus \{0\}$, this contradicts polystability. ∎

5 Examples and Applications

Our main theorem generalize previous results on "universal" Kobayashi-Hitchin correspondences in two directions

1. It holds when the base manifold is an arbitrary Hermitian manifold,

2. It holds for *oriented* pairs, giving the appropriate *complex geometric* stability and polystability conditions for this class of objects.

In this chapter we will illustrate in concrete situations the importance of both directions. Using the correspondence between polystable oriented pairs and solutions of the generalized Hermitian-Einstein equations, we construct a canonical isomorphism between the corresponding moduli spaces, the *Kobayashi-Hitchin isomorphism*. Using this isomorphism and ideas from Donaldson theory, we endow the moduli spaces with canonical Hermitian metrics. We discuss in more detail moduli spaces of oriented connections, Douady Quot-spaces and moduli spaces of non-abelian monopoles in the non-Käherian framework.

5.1 Oriented holomorphic principal bundles and oriented connections

5.1.1 The Kobayashi-Hitchin correspondence for oriented holomorphic principal bundles

Let $\hat{Q}$ be a principal $\hat{G}$ bundle over a compact complex manifold and $\mathcal{Q}_0$ a fixed bundle holomorphic structure of the G-quotient Q_0. Our moduli problem asks:

Classify bundle holomorphic structures $\hat{\mathcal{Q}}$ on $\hat{Q}$ which induce $\mathcal{Q}_0$ on Q_0, modulo the gauge group $\mathcal{G} := \mathrm{Aut}_{Q_0}(\hat{Q}) = \Gamma(\hat{Q} \times_{\mathrm{Ad}} G)$.

Such a holomorphic structure will be naturally called a $\mathcal{Q}_0$-*oriented* holomorphic structure on $\hat{Q}$

Example: Choosing $\hat{G} = GL(r, \mathbb{C})$, $G = SL(r)$ our problem becomes: Let E be a differentiable vector bundle vector bundle over X and $\mathcal{L}$ a holomorphic structure on its determinant line bundle L. Classify all holomorphic structures $\mathcal{E}$ on E such that $\det(\mathcal{E}) = \mathcal{L}$ modulo the gauge group $\mathcal{G} := \Gamma(X, SL(E))$.

Note first that the $\mathcal{G}$-stabilizer of a holomorphic structure $\hat{\mathcal{Q}}$ always contains the relative centralizer $Z_G(\hat{G}) = \{g \in G | \mathrm{Ad}_\gamma(g) = g \ \forall \gamma \in \hat{G}\}$, so it is natural to relax the *simplicity condition* for an oriented holomorphic structure $\hat{\mathcal{Q}}$ by asking its infinitesimal stabilizer to be minimal, i.e. $\mathfrak{g}_{\hat{\mathcal{Q}}} = z(\mathfrak{g}, \hat{G})$, (see Definition 2.19).

Our bundle classification problem can be formally regarded as a coupled problem with fibre $F = \{*\}$. Let T be the unique maximal compact subgroup of the abelian reductive complex group $Z_G(\hat{G})$ and $\mathfrak{t} := \mathrm{Lie}(T)$. Let h be an ad-invariant complex inner product of Euclidean type on $\hat{\mathfrak{g}}$. The choice of a Hamiltonian system $(K \subset \hat{K}, g_F, \mu)$ with a $\hat{K}$-equivariant moment map μ reduces to just the choice of an element $it \in \mathfrak{t}$. Therefore, our stability condition becomes

A $\mathcal{Q}_0$-oriented holomorphic structure $\hat{\mathcal{Q}}$ on $\hat{Q}$ is t-stable if its infinitesimal stabilizer $\mathfrak{g}_{\hat{\mathcal{Q}}}$ coincides with $z(\mathfrak{g}, \hat{G})$ and the following holds: for every vector $\xi \in H(G)$ and meromorphic $\hat{G}(\xi)$-reduction ρ of $\hat{\mathcal{Q}}$, one has

$$\frac{2\pi}{(n-1)!} \deg(\rho, h(\xi)) \geq h(t, \xi) Vol_g(X) \tag{67}$$

with strict inequality when $\xi \notin z(\mathfrak{g}, \hat{G})$.

A vector $\xi \in H(G)$ decomposes as $\xi = \xi_0 + \zeta$, where $\xi_0 \in z(\mathfrak{g}, \hat{G})^{\perp}$ and $\zeta \in z(\mathfrak{g}, \hat{G})$. The linear form $h(\zeta)$ is $\mathrm{ad}_{\hat{G}}$-invariant, so $\deg(\rho, h(\zeta))$ can be identified with a *holomorphic invariant* $\deg(\hat{\mathcal{Q}}, h(\zeta))$ of $\hat{\mathcal{Q}}$ (see Definition 2.8). Since h is non-degenerate, there exists a unique element $\tau(\hat{\mathcal{Q}}) \in i\mathfrak{t}$ such that

$$\deg(\hat{\mathcal{Q}}, h(\zeta)) = h(\tau(\hat{\mathcal{Q}}), \zeta) \ \forall \zeta \in i\mathfrak{t} = H(Z_G(\hat{G})) \ .$$

The holomorphic invariant τ plays the role of the degree in the classical case of (non-oriented) bundles; it is obviously a topological invariant when g is Kählerian. Since the map, $\zeta \mapsto \deg(\hat{\mathcal{Q}}, h(\zeta))$ is linear on $i\mathfrak{t}$, the inequality (67) implies

$$t = \frac{2\pi\tau(\hat{\mathcal{Q}})}{(n-1)! Vol_g(X)} \ .$$

which means that *an oriented holomorphic structure $\hat{\mathcal{Q}}$ can be t-stable only for a single constant t, which is a holomorphic invariant of $\hat{\mathcal{Q}}$ (and a topological invariant of $\hat{Q}$ when g is Kähler).* This motivates the following

Definition 5.1 *A $\mathcal{Q}_0$-oriented holomorphic structure $\hat{\mathcal{Q}}$ on $\hat{Q}$ is called:*

1. *semistable, if $\deg(\rho, h(\xi)) \geq 0$ for every vector $\xi \in H(G) \cap z(\mathfrak{g}, \hat{G})^{\perp}$ and meromorphic reduction ρ of $\hat{\mathcal{Q}}$ to $\hat{G}(\xi)$,*

2. *stable, if it is semistable, its infinitesimal stabilizer $\mathfrak{g}_{\hat{\mathcal{Q}}}$ coincides with $z(\mathfrak{g}, \hat{G})$ and the semistability inequality above is strict when $\xi \neq 0$,*

3. *polystable, if it is semistable, its infinitesimal stabilizer $\mathfrak{g}_{\hat{\mathcal{Q}}}$ is reductive in the sense of Definition 2.16, and the semistability inequality is equality*

when and only when ρ is induced by a global holomorphic $Z_{\hat{G}}(\xi)$-reduction and the section in $H^0(\hat{\mathcal{Q}} \times_{\mathrm{ad}} \mathfrak{g})$ defined by ξ via this reduction belongs to $\mathfrak{g}_{\hat{\mathcal{Q}}}$.

The condition $\xi \in \mathfrak{z}^{\perp}$ becomes simply "ξ is semisimple" when $G = \hat{G}$ is connected.

Remark 5.2 *Using the fact that, for a fixed parabolic subgroup $L \subset \hat{G}$ and maximal compact subgroup $K \subset G$, the map $\xi \mapsto \deg(\rho, h(\xi))$ is affine on the convex set $\{s \in i\mathfrak{k} | \hat{G}(s) \supset L\}$, it follows easily that it suffices to check the conditions above only for vectors ξ, for which $\hat{G}(\xi)$ is maximal in the set*

$$\mathcal{P}ar_G(\hat{G}) := \{\hat{G}(\xi) | \xi \in H(G)\} .$$

In this way, one recovers the usual (semi, poly) - stability condition for vector bundles in the classical cases $(G, \hat{G}) = (GL(r, \mathbb{C}), GL(r, \mathbb{C}))$, $(G, \hat{G}) = (SL(r, \mathbb{C}), GL(r, \mathbb{C}))$.

Our main theorem becomes

Theorem 5.3 *Fix any K_0 reduction $P_0 \subset Q_0$. A $\mathcal{Q}_0$-oriented holomorphic structure $\hat{\mathcal{Q}}$ on $\hat{Q}$ is polystable if and only if there exists a $\hat{K}$ reduction $\hat{P}$ of $\hat{Q}$ inducing P_0 on Q_0 such that the Chern connection $\hat{A}$ of the pair $(\hat{\mathcal{Q}}, \hat{P})$ satisfies the Hermitian-Einstein equation*

$$\mathrm{pr}_{i\mathfrak{k}}[i\Lambda_g F_{\hat{A}}] = \frac{2\pi\tau(\hat{\mathcal{Q}})}{(n-1)!Vol_g(X)} . \tag{68}$$

Note again that the right hand term (the "Einstein constant") of this equation is a topological invariant of $\hat{Q}$ in the Kählerian case and a holomorphic invariant of $\hat{\mathcal{Q}}$ in the Gauduchon case.

5.1.2 Isomorphism of moduli spaces

As explained in the introduction, a Kobayashi-Hitchin correspondence always yields an isomorphism between a moduli space of solutions of a Hermitian-Einstein type equation (which is a very difficult non-linear PDE system) and a moduli stable of polystable objects, which in many cases can be described with complex geometric methods. This idea was first used in Donaldson theory with spectacular success.

The formalism explained in [DK] and [LT] for comparing moduli spaces can be easily extended for general oriented bundles as follows.

Let $\bar{\mathcal{A}} := \bar{\mathcal{A}}_{J_0}(\hat{Q})$ be the space of oriented almost complex structures $\hat{J}$ on $\hat{Q}$ which project onto J_0 (the almost complex structure of $\mathcal{Q}_0$). $\bar{\mathcal{A}}$ is an affine space with $A^{0,1}(\hat{Q} \times_{\mathrm{ad}} \mathfrak{g})$ as model vector space.

Let $\mathcal{H} = \mathcal{H}_{J_0}(\hat{Q}) \subset \bar{\mathcal{A}}$ be the subset of integrable almost complex structures, $\mathcal{H}^{\mathrm{s}}$ the open subset of simple structures and $\mathcal{H}_g^{\mathrm{st}} \subset \mathcal{H}^{\mathrm{s}}$ the subspace of stable structures.

On the differential geometric side we choose a $\hat{K}$-reduction $\hat{P}$ of $\hat{Q}$, we denote by P_0 the induced K_0-reduction of Q_0 and by A_0 the Chern connection of the pair (P_0, J_0). Let $\mathcal{A} := \mathcal{A}_{A_0}(\hat{P})$ be the space of A_0-*oriented connections*, i.e. the space of connections on $\hat{P}$ which project onto the fixed connection A_0. A connection $\hat{A} \in \mathcal{A}$ will be called *irreducible* if its infinitesimal stabilizer with respect to the $\mathcal{K}$-action is minimal, i.e. it coincides with the centralizer $z(\mathfrak{k}, \hat{K})$ of $\hat{K}$ in $\mathfrak{k}$. As usually, we will use the superscript $(\cdot)^*$ to denote the open irreducible part and the superscript $(\cdot)^{\mathrm{HE}}$ to denote the Hermitian-Einstein part, i.e. the locus of *integrable* solutions of equation (68).

One can introduce a stronger irreducibility condition as follows: a connection $\hat{A} \in \mathcal{A}$ is called *strongly irreducible* if its stabilizer $\mathcal{K}_A \subset \mathcal{K}$ is minimal, i.e. it coincides with the centralizer $Z_K(\hat{K})$. Since this condition is not easily checked, and since we want to avoid unimportant technical complications, we will impose a natural condition on our pair of Lie groups $(K, \hat{K})$ which assures that the two irreducibility conditions coincide, and which holds for the standard pairs of classical groups. One can easily show

Proposition 5.4 *Suppose that K is connected and that for every $k \in K$ there exists $\lambda \in \mathfrak{k}$ which is invariant under $Z_{\hat{K}}(k)$ and such that $\exp(\lambda) = k$. Then every irreducible connection $\hat{A} \in \mathcal{A}$ is strongly irreducible.*

For the remainder of this section, we will always assume that the pair $(K, \hat{K})$ satisfies the hypothesis of this proposition.

The fundamental objects we study in this section are the moduli spaces

$$\mathcal{M}^{\mathrm{s}} := {}^{\mathcal{H}^{\mathrm{s}}}\!/_{\mathcal{G}} \ , \ \mathcal{M}_g^{\mathrm{st}} := {}^{\mathcal{H}_g^{\mathrm{st}}}\!/_{\mathcal{G}} \ , [\mathcal{M}^{HE}]^* := {}^{[\mathcal{A}^{\mathrm{HE}}]^*}\!/_{\mathcal{K}} \ .$$

The same methods as in [Do1], [LT] show that Theorem 5.3 can be refined and that one has a canonical isomorphism of moduli spaces:

Theorem 5.5 *1. $\mathcal{M}^{\mathrm{s}}$ has a natural structure of a complex analytic space (in general non-Hausdorff) of finite dimension and $[\mathcal{M}^{HE}]^*$ has a natural structure of a Hausdorff real analytic space of finite dimension.*

2. The Chern map $\hat{A} \mapsto J^{\hat{A}}$ induces a real analytic open embedding

$$[\mathcal{M}^{HE}]^* \xrightarrow{\mathrm{KH}} \mathcal{M}^{\mathrm{s}}$$

whose image is $\mathcal{M}_g^{\mathrm{st}}$. In particular, this space is open in $\mathcal{M}^{\mathrm{s}}$, is Hausdorff and is canonically isomorphic via KH to $[\mathcal{M}^{HE}]^*$.

Remark 5.6 *1. In general, the moduli space $[\mathcal{M}^{\mathrm{HE}}]^*$ can contain orbifold type singularities if it contains points which are not strongly irreducible. Our condition on the pair $(K, \hat{K})$ avoids this complication.*

2. One can show that the isomorphism $[\mathcal{M}^{HE}]^* \xrightarrow{KH} \mathcal{M}^{\mathrm{st}}$ extends to a homeomorphism $[\mathcal{M}^{HE}] \xrightarrow{KH} \mathcal{M}_g^{\mathrm{pst}}$, where

$$\mathcal{M}_g^{\mathrm{pst}} := \mathcal{H}_g^{\mathrm{pst}}\big/\mathcal{G}$$

is the moduli space of polystable holomorphic structures. This space, in general, does not have a complex space structure; moreover the local structure around a non-stable polystable point can be very complicated (see e.g. [Te4]).

Why are moduli spaces of oriented connections important? A clear, simple motivation for considering moduli spaces of *oriented* connections is the following: extending the Witten conjecture to non-simply connected 4-manifolds leads naturally to Donaldson moduli spaces of oriented, projectively ASD $U(2)$-connections. This spaces cannot be identified in general with moduli spaces of $PU(2)$-connections. We explain in detail this phenomenon:

Let E be a Hermitian 2-bundle on an oriented Riemannian 4-manifold M, P_E its frame bundle, and L its determinant line bundle. Fix a unitary connection $A_0 \in \mathcal{A}(L)$, denote by $\mathcal{A}_{A_0}$ the space of A_0-oriented connections, i.e. the space of unitary connections on E which induce A_0 on L. Consider the subspace

$$\mathcal{A}_{A_0}^{\mathrm{ASD}} := \{A \in \mathcal{A}_{A_0}(E) \mid (F_A^0)_+ = 0\}$$

of *projectively ASD oriented connections*. Here F_A^0 stands for the trace-free part of the curvature, which can be identified with the curvature of the induced $PU(2)$-connection $\bar{A}$ on the associated $PU(2)$-bundle $\bar{P} := P_E\big/S^1$. Consider the moduli space

$$\mathcal{M}_{A_0}^{\mathrm{ASD}}(E) := \mathcal{A}_{A_0}^{\mathrm{ASD}}\big/\mathcal{K}$$

of projectively ASD oriented connections on E modulo the gauge group

$$\mathcal{K} := \Gamma(X, SU(E)) \ .$$

The space $\mathcal{A}_{A_0}^{\mathrm{ASD}}(E)$ can be identified with the space $\mathcal{A}^{\mathrm{ASD}}(\bar{P})$ of ASD connections on $\hat{P}$, but the natural map $\mathcal{K} \to \mathrm{Aut}(\bar{P})$ between the two gauge groups is in general not surjective. There is a natural map

$$\mathcal{M}_{A_0}^{\mathrm{ASD}}(E) \longrightarrow \mathcal{M}^{ASD}(\bar{P})$$

which is always surjective and is an isomorphism when $H^1(M, \mathbb{Z}_2) = 0$. In the general case, one has a natural action of $H^1(M, \mathbb{Z}_2)$ on $\mathcal{M}_{A_0}^{\mathrm{ASD}}(E)$ given by tensorizing with flat $\{\pm 1\}$-connections, and $\mathcal{M}^{\mathrm{ASD}}(\hat{P})$ is just the $H^1(M, \mathbb{Z}_2)$-quotient ([LT], [Te4]). This natural $H^1(M, \mathbb{Z}_2)$-action might have fixed points [Te4], so the space $\mathcal{M}^{ASD}(\bar{P})$ is in general more singular than $\mathcal{M}_{A_0}^{\mathrm{ASD}}(E)$. But there is another important reason why the latter space is a more convenient object: in order to embed a Donaldson moduli space in a moduli space of non-abelian monopoles (this is the starting point of the "cobordism strategy" for the proof of the Witten conjecture), one needs coupled Dirac operators (see section 5.3.2), so one needs unitary connections. There is no obvious way to define a coupled Dirac operator associated with a $PU(2)$-connection.

5.1.3 The canonical Hermitian metric on moduli spaces of oriented connections

We write Z and z for $Z_K(\hat{K})$ and $z(\mathfrak{k}, \hat{K})$ respectively. Let

$$\mathcal{K}_r := \mathcal{K}/Z \ , \ A_r^0(\hat{P} \times_{\mathrm{ad}} \mathfrak{k}) := z^{\perp} \subset A^0(\hat{P} \times_{\mathrm{ad}} \mathfrak{k}) = \mathrm{Lie}(\mathcal{K})$$

be the *reduced gauge group* and its Lie algebra. We will also need the orthogonal projection

$$p_r : A^0(\hat{P} \times_{\mathrm{ad}} \mathfrak{k}) \longrightarrow A_r^0(\hat{P} \times_{\mathrm{ad}} \mathfrak{k}) \ .$$

This group acts freely on our configuration space of oriented irreducible connections $\mathcal{A}^*$ from the right, so $\mathcal{A}^*$ can be regarded as a principal $\mathcal{K}_r$-bundle over the quotient $\mathcal{B}^* := \mathcal{A}^*/\mathcal{K}_r$.

Write $\hat{K}_r := \hat{K}/Z$ and consider the principal $\hat{K}_r$-bundle

$$\hat{P}_r := \hat{P}/Z := \hat{P} \times_{\hat{K}} \hat{K}_r \ .$$

The associated bundle

$$\hat{\mathbb{P}}_r := \mathcal{A}^* \times_{\mathcal{K}_r} \hat{P}_r$$

with standard fibre $\hat{P}_r$ over $\mathcal{B}^*$ can be alternatively regarded as a principal $\hat{K}_r$-bundle over $\mathcal{B}^* \times X$ and will be called the *universal $\hat{K}_r$-bundle*.

We will introduce a canonical connection on this principal bundle (compare with [DK], section 5.2). Since our base metric is not Kähler (but only Gauduchon) one should not use the L^2-orthogonal distribution used in classical gauge theory (see [DK] p. 196).

Following the strategy in [LT] section 5.3 we put, for any $A \in \mathcal{A}^*$,

$$\delta_{\hat{A}} := \Lambda_g \circ d_{\hat{A}}^c : A^1(\hat{P} \times_{\mathrm{ad}} \mathfrak{k}) \longrightarrow A^0(\hat{P} \times_{\mathrm{ad}} \mathfrak{k}) \ ,$$

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$$q_{\hat{A}} := p_r \circ \delta_{\hat{A}} \circ d_{\hat{A}} : A_r^0(\hat{P} \times_{\mathrm{ad}} \mathfrak{k}) \to A_r^0(\hat{P} \times_{\mathrm{ad}} \mathfrak{k}) \ .$$

The operator $\delta_{\hat{A}}$ plays the role of the L^2-adjoint of $d_{\hat{A}}$ in the Kählerian framework, whereas $q_{\hat{A}}$ replaces the standard Laplacian.

The same methods as in [LT] show that

Proposition 5.7 *After suitable Sobolev completions the following holds: The assignment*

$$\hat{A} \mapsto \ker(p_r \circ \delta_{\hat{A}})$$

defines a real analytic connection Γ in the $\mathcal{K}_r$-principal bundle $\mathcal{A}^* \to \mathcal{B}^*$.

Proof: We recall briefly the argument in [LT] in this more general framework. First note that, in general, for any Euclidean connection B on a Euclidean real bundle W over X, the following identity holds pointwise:

$$\frac{1}{2}\Lambda_g d^c d|v|^2 = (\Lambda_g d_B^c d_B v, v) - |d_B v|^2 \tag{69}$$

We show first that

$$\ker(p_r \circ \delta_{\hat{A}}) \cap T_{\hat{A}}[\hat{A} \cdot \mathcal{K}_r] = 0 \tag{70}$$

for every connection $\hat{A} \in \mathcal{A}^*$. Indeed, for a an element $u \in A_r^0(\hat{P} \times_{\mathrm{ad}} \mathfrak{k}) = \mathrm{Lie}(\mathcal{K}_r)$, one has

$$p_r \delta_{\hat{A}} d_{\hat{A}} u = p_r \Lambda_g d_{\hat{A}}^c d_{\hat{A}} u \ .$$

The vanishing of this implies $\Lambda_g d_{\hat{A}}^c d_{\hat{A}} u = \zeta \in z$, so

$$\frac{1}{2}\Lambda d^c d|u|^2 + |d_{\hat{A}} u|^2 = (\Lambda_g d_{\hat{A}}^c d_{\hat{A}} u, u) = (\zeta, u) \ .$$

Integrating on X and taking into account that g is a Gauduchon metric, one gets $\|d_{\hat{A}} u\|_{L^2}^2 = 0$. Since our oriented connection is irreducible, this implies $u = 0$, so (70) holds. But this argument also shows that the operator $q_{\hat{A}}$ is injective; but it is easy to see that it is an index 0 Fredholm operator, so it is also surjective. Any $\alpha \in A^1(\hat{P} \times_{\mathrm{ad}} \mathfrak{k})$ can be written as

$$\alpha = d_{\hat{A}}(q_{\hat{A}}^{-1}(p_r \delta_{\hat{A}}(\alpha))) + (\alpha - d_{\hat{A}}(q_{\hat{A}}^{-1}(p_r \delta_{\hat{A}}(\alpha)))) \ ,$$

and it is obvious that the first term belongs to $T_{\hat{A}}[\hat{A} \cdot \mathcal{K}_r]$, whereas the second to $\ker(p_r \circ \delta_{\hat{A}})$. Therefore, Γ is a horizontal distribution, and it is easily seen to be $\mathcal{K}_r$-invariant. ∎

Remark 5.8 *The curvature F_Γ of the connection Γ is given by the formula*

$$F_\Gamma(\alpha, \beta) = -q_{\hat{A}}^{-1} p_r \Lambda_g [\alpha^c \wedge \beta] \ .$$

for Γ-horizontal vectors α, β.

(see [LT] p. 177). Note that $\Lambda_g[\alpha^c \wedge \beta]$ is obtained by coupling the inner product g on 1-forms with the Lie bracket on the Lie algebra $\mathfrak{k}$. Therefore it coincides with the term $\{a, b\}$ appearing in [DK].

The connection Γ can be used to define a *canonical Hermitian metric H* on the smooth part $[\mathcal{M}^{\text{HE}}]^*_{\text{reg}}$ of the moduli space $[\mathcal{M}^{\text{HE}}]^* \simeq \mathcal{M}^{\text{st}}_g$. Recall that we have fixed an ad-invariant complex inner product of Euclidean type h on the complex Lie algebra $\hat{\mathfrak{g}}$. We put

$$\Omega(a, b) := -\int_X h(\alpha \wedge \beta) \wedge \omega_g^{n-1} \ ,$$

where a, b are tangent vectors to the moduli space and α, β are Γ-horizontal lifts. This form has type $(1, 1)$ with respect to the complex structure induced from the canonical complex structure $\mathcal{J}$ of $A^1(\hat{P} \times_{\text{ad}} \mathfrak{k}) \simeq A^{0,1}(\hat{Q} \times_{\text{ad}} \mathfrak{g})$ on the $\mathcal{J}$-invariant spaces $\Gamma_{\hat{A}} \cap T_{\hat{A}}(\mathcal{A}^{\text{HE}})$. It is also easy to see that this form is positive with respect to $\mathcal{J}$, so it defines a Hermitian metric H as claimed.

Note that using the standard L^2-orthogonal distribution (as in [DK] section 5.3) will in general only yield a Riemannian metric, not a Hermitian one when g is not Kähler.

On the product $\mathcal{A}^* \times \hat{P}_r$ we define now a distribution D which is horizontal with respect to the projection $\mathcal{A}^* \times \hat{P}_r \to \mathcal{B}^* \times X$. We put

$$D_{(\hat{A},p)} := \Gamma_{\hat{A}} \times \hat{A}_p \subset T_{(\hat{A},p)}(\mathcal{A}^* \times \hat{P}_r) \ ,$$

where $\Gamma_{\hat{A}}$ is the horizontal space of Γ at $\hat{A}$ and $\hat{A}_p$ is the horizontal space at p of the connection $\hat{A}_r$ induced by $\hat{A}$ on $\hat{P}_r$. This distribution is invariant with respect to the diagonal $\hat{\mathcal{K}}_r$-action and the right $\hat{K}_r$-action on $\hat{P}_r$, hence it descends to a connection $\mathbb{A}$ on the universal $\hat{K}_r$-bundle $\hat{\mathbb{P}}_r$.

The curvature $\mathbb{F}$ of this connection is (see [DK], p. 197 and [LT] p. 177 for the Gauduchon case):

$$\mathbb{F}_{[\hat{A},p]}(u, v) = F_{\hat{A}_r,p}(u, v) \ , \ \mathbb{F}_{[\hat{A},p]}(\alpha, v) = a(v) \ , \ \mathbb{F}_{[\hat{A},p]}(\alpha, \beta) = F_\Gamma(\alpha, \beta) \quad (71)$$

for tangent vectors u, $v \in T_x(X)$ and α, $\beta \in \Gamma_{\hat{A}}$.

Remark 5.9 *The restriction of $\mathbb{A}$ to $\mathbb{P}\big|_{[\mathcal{M}^{\text{HE}}]^*_{\text{reg}} \times X}$ is an integrable connection.*

Using our ad-invariant inner product h, we get an associated *closed characteristic form* $h(\mathbb{F}, \mathbb{F}) \in A^4(\mathcal{B}^* \times X)$.

For any $\nu \in A^{2n-2}(X)$ and a, $b \in T_{[\hat{A}]}(\mathcal{B}^*)$ we get as in [DK] p. 179

$$[p_{\mathcal{B}*}]_*[h(\mathbb{F}, \mathbb{F}) \wedge p_X^*(\nu)](a, b) = \int_X \left[-h(\alpha \wedge \beta) + 2h\left(F_{\hat{A}_r}, F_\Gamma(\alpha, \beta)\right) \right] \wedge \nu$$

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$$= \int_X \left[-h(\alpha \wedge \beta) + 2h\left(\mathrm{pr}_{\mathfrak{k}}(F_{\hat{A}_r}), F_\Gamma(\alpha, \beta) \right) \right] \wedge \nu \ ,$$

where α $\beta \in \Gamma_{\hat{A}}$ are horizontal lifts of a and b at $\hat{A}$, $p_{\mathcal{B}^*}$, p_X stand for the respective projections, and $[p_{\mathcal{B}^*}]_*$ denotes fibre integration along the X-fibres.

Therefore, taking $\nu = \omega_g^{n-1}$ and using the Hermitian-Einstein equation (68), on the smooth part $[\mathcal{M}^{\mathrm{HE}}]_{\mathrm{reg}}^*$ of the moduli space it holds

$$[p_{\mathcal{B}^*}]_*[h(\mathbb{F}, \mathbb{F}) \wedge p_X^*(\omega_g^{n-1})]\Big|_{[\mathcal{M}^{\mathrm{HE}}]_{\mathrm{reg}}^*} = \Omega \tag{72}$$

Taking into account that $h(\mathbb{F}, \mathbb{F})\Big|_{[\mathcal{M}^{\mathrm{HE}}]_{\mathrm{reg}}^* \times X}$ is a closed $(2, 2)$ form (see Remark 5.9), we get

Theorem 5.10 *The $(1, 1)$-form Ω associated with the canonical Hermitian metric H on the smooth part $[\mathcal{M}_g^{\mathrm{st}}]_{\mathrm{reg}}$ of the moduli space $\mathcal{M}_g^{\mathrm{st}}$ has the following property:*

1. *it is Kähler, when $d(\omega_g^{n-1}) = 0$ (i.e. when g is semi-Kähler),*

2. *it satisfies the identity $\partial \bar{\partial} \Omega = 0$ when g is Gauduchon.*

Note that, for the second statement, one needs that $h(\mathbb{F}, \mathbb{F})\Big|_{[\mathcal{M}^{\mathrm{HE}}]_{\mathrm{reg}}^* \times X}$ is ∂ and $\bar{\partial}$-closed, so it is important to know that this form has pure type $(2, 2)$.

This result shows that the moduli spaces of stable oriented bundles over a compact complex manifold have interesting differential geometric properties (see Remark 5.19 below).

5.2 Moduli spaces of oriented pairs

5.2.1 Isomorphy of moduli spaces

The construction of the moduli spaces of (oriented) holomorphic pairs in the non-linear case (when the action $\hat{\alpha}$ is not a linear representation) is substantially more difficult than in the classical moduli theories.

Indeed, in the non-linear case, it is quite difficult to construct the Sobolev completions of the space of sections in the associated bundle $E := \hat{Q} \times_{\hat{G}} F$ and to endow these completions with the structure of a Banach manifold (see [Pa]).

Since here we are only interested in geometric applications, we will skip these analytical difficulties, which will be addressed in future work.

Fixing a holomorphic structure J_0 on Q_0, our complex geometric configuration space is
$$\bar{\mathcal{A}} := \bar{\mathcal{A}}_{J_0}(\hat{Q}) \times \Gamma(X, E)$$
with the natural right action of the complex gauge group $\mathcal{G} = \Gamma(X, \hat{Q} \times_{\mathrm{Ad}} G)$.

Let $\mathcal{H} \subset \bar{\mathcal{A}}$ be the subspace of integrable pairs, $\mathcal{H}^{\mathrm{s}}$ the open subset of *strongly simple* integrable pairs, i.e. of pairs with trivial stabilizer. Note that the stabilizer of a simple integrable pair can be in general non-trivial; in order to avoid complications caused by orbifold-type singularities in the moduli spaces, we will restrict ourselves to the strongly simple points. Similarly, $\bar{\mathcal{A}}^{\mathrm{s}}$ will denote the subspace of $\bar{\mathcal{A}}$ consisting of pairs with trivial stabilizer. The quotient
$$\mathcal{M}^{\mathrm{s}} := {}^{\mathcal{H}^{\mathrm{s}}}\!/\!_{\mathcal{G}}$$
has the structure of a finite dimensional, in general non-Hausdorff complex space, which will be called the moduli space of strongly simple oriented holomorphic pairs. This space can be regarded as the vanishing locus of a section in a vector bundle over $\bar{\mathcal{B}}^{\mathrm{s}} := {}^{\bar{\mathcal{A}}^{\mathrm{s}}}\!/\!_{\mathcal{G}}$ whose fibre over $[\hat{J}, \varphi]$ is
$$A^{0,2}(\hat{Q} \times_{\mathrm{ad}} \mathfrak{g}) \times A^{0,1}(\varphi^*(V)) \ ,$$
where $V \to E$ is the vertical tangent bundle of E. This section is induced by the left hand side of the integrability condition.

Fix now a $\hat{K}$-reduction $\hat{P}$ of $\hat{Q}$, let P_0 be the induced reduction of Q_0 and A_0 the Chern connection of the pair (P_0, J_0). The configuration space is now
$$\mathcal{A} := \mathcal{A}_{A_0}(\hat{P}) \times \Gamma(X, E)$$
acted on by the gauge group $\mathcal{K} := \Gamma(\hat{P} \times_{\mathrm{ad}} K)$. We will indicate by ( )* the respective parts of strongly irreducible pairs (i.e. of pairs with trivial stabilizer), and by ( )$^{\mathrm{HE}}$ the subspaces of integrable pairs solving the generalized Hermitian-Einstein equation
$$\mathrm{pr}_{i\mathfrak{k}}\left[i\Lambda_g F_{\hat{A}}\right] + i\mu(\varphi) = 0 \ .$$

We are interested in the moduli spaces
$$\mathcal{M}^{\mathrm{HE}} := {}^{\mathcal{A}^{\mathrm{HE}}}\!/\!_{\mathcal{K}} \ , \ [\mathcal{M}^{\mathrm{HE}}]^* := {}^{[\mathcal{A}^{\mathrm{HE}}]^*}\!/\!_{\mathcal{K}} \ .$$
Both moduli spaces are Hausdorff, and the second moduli space can be described as the vanishing locus of a section s in a bundle S over the infinite dimensional quotient $\mathcal{B}^* := {}^{\mathcal{A}^*}\!/\!_{\mathcal{K}}$. Note however, that this space has no natural real-analytic structure in the general non-linear framework (even when μ is a real analytic map on F). The point is that the bundle S will only be a $\mathcal{C}^k$-bundle, for a regularity class k depending on the chosen Sobolev completion of $\Gamma(X, E)$.

The Kobayashi-Hitchin correspondence becomes

Theorem 5.11   *1. The map $(\hat{A}, \varphi) \mapsto (J^{\hat{A}}, \varphi)$ induces an open embedding*

$$[\mathcal{M}^{HE}]^* \xrightarrow{\ KH\ } \mathcal{M}^{\mathrm{s}}$$

whose image is $\mathcal{M}_g^{\mathrm{st}}$. In particular, this space is open in $\mathcal{M}^{\mathrm{s}}$, is Hausdorff and is canonically isomorphic via KH to $[\mathcal{M}^{HE}]^*$.

As in the bundle case, it holds:

Remark 5.12 *The isomorphism $[\mathcal{M}^{HE}]^* \xrightarrow{\ KH\ } \mathcal{M}^{\mathrm{st}}$ extends to a homeomorphism $[\mathcal{M}^{HE}] \xrightarrow{\ KH\ } \mathcal{M}_g^{\mathrm{pst}}$, where $\mathcal{M}_g^{\mathrm{pst}}$ is the moduli space of polystable oriented holomorphic pairs.*

5.2.2   The canonical Hermitian metric on moduli spaces of oriented pairs

The canonical Hermitian metric on the moduli space.   As in section 5.1.3, one can define a natural connection on the principal $\mathcal{K}$-bundle $\mathcal{A}^* \to \mathcal{B}^*$. Let E be the associated bundle $E := \hat{P} \times_{\hat{\alpha}} F$ and $V \to E$ the vertical tangent bundle of E. In this section we will always assume that the Kählerian structure (J_F, ω_F) on F is $\hat{K}$-invariant. We will denote by the same symbols the induced complex, respectively symplectic structure on the fibres of E and V.

The tangent space of $\mathcal{A}^*$ at a pair $\mathfrak{a} = (\hat{A}, \varphi)$ is the product $A^1(\hat{P} \times_{\mathrm{ad}} \mathfrak{k}) \times A^0(\varphi^*(V))$, whereas the tangent space to the orbit $\mathfrak{a} \cdot \mathcal{K}$ is $T_{\mathfrak{a}}(\mathfrak{a} \cdot \mathcal{K}) = \mathrm{im}(\mathfrak{d}_{\mathfrak{a}})$ for the operator

$$\mathfrak{d}_{\mathfrak{a}} : A^0(\hat{P} \times_{\mathrm{ad}} \mathfrak{k}) \to A^1(\hat{P} \times_{\mathrm{ad}} \mathfrak{k}) \times A^0(\varphi^*(V)) \ , \ \ \mathfrak{d}_{\mathfrak{a}}(k) := (d_{\hat{A}} k, \varphi^*(k^\#)) \ ,$$

where $k^\#$ denotes the section of V defined by k.

We define for $\mathfrak{a} = (\hat{A}, \varphi) \in \mathcal{A}^*$ the operators

$$\delta_{\mathfrak{a}} : A^1(\hat{P} \times_{\mathrm{ad}} \mathfrak{k}) \times A^0(\varphi^*(V)) \longrightarrow A^0(\hat{P} \times_{\mathrm{ad}} \mathfrak{k}) \ , \ \delta_{\mathfrak{a}}(\alpha, \psi) = \Lambda_g d_{\hat{A}}^c(\alpha) + d_\varphi \mu(J_F \psi) \ ,$$

where in the right hand term, μ is regarded as a map $\Gamma(X, E) \to A^0(\hat{P} \times_{\mathrm{ad}} \mathfrak{k})$ and d_φ is the differential of this map at φ, and

$$q_{\mathfrak{a}} : A^0(\hat{P} \times_{\mathrm{ad}} \mathfrak{k}) \longrightarrow A^0(\hat{P} \times_{\mathrm{ad}} \mathfrak{k}) \ , \ q_{\mathfrak{a}} := \delta_{\mathfrak{a}} \circ \mathfrak{d}_{\mathfrak{a}} \ .$$

The precise result is the following:

Proposition 5.13 *The assignment $\mathfrak{a} \mapsto \ker(\delta_{\mathfrak{a}})$ defines a connection Γ on the principal $\mathcal{K}$-bundle $\mathcal{A}^* \to \mathcal{B}^*$.*

Proof: As in the proof of Proposition 5.7 it suffices to show that the operator $q_{\mathfrak{a}}$ is injective. Using the fact μ can be regarded as a moment map for the $\mathcal{K}$-action on the space of sections $\Gamma(X, E)$, we get pointwise in X

$$(q_{\mathfrak{a}}(k), k) = (\Lambda_g d_{\hat{A}}^c d_{\hat{A}} k + d_\varphi \mu(J_F \varphi^*(k^\#)), k) =$$

$$= (\Lambda_g d_{\hat{A}}^c d_{\hat{A}} k, k) + \omega_F(\varphi^*(k^\#), J_F \varphi^*(k^\#)) = \frac{1}{2} \Lambda_g d^c d |k|^2 + |d_{\hat{A}} k|^2 + |\varphi^*(k^\#)|^2 \ .$$

Hence, if $q_{\mathfrak{a}}(k) = 0$, then integrating on X we get $d_{\hat{A}} k = 0$, $\varphi^*(k^\#) = 0$, so $k = 0$, because the oriented pair $(\hat{A}, \varphi)$ is irreducible. $\blacksquare$

Again this connection can be used to define a canonical Hermitian metric H on the smooth part of the moduli space $[\mathcal{M}^{\mathrm{HE}}]^* \simeq \mathcal{M}_g^{\mathrm{st}}$. We put

$$\Omega(a, b) := -\int_X h(\alpha \wedge \beta) \wedge \omega_g^{n-1} + (n-1)! \int_X \omega_F(\psi, \chi) vol_g,$$

where a, b are tangent vectors to the moduli space and (α, ψ), (β, χ) are Γ-horizontal lifts at pair $\mathfrak{a} = (\hat{A}, \varphi)$. Note again that our definition of the connection Γ implies that this form has type $(1, 1)$ with respect to the complex structure induced by the natural complex structure $\mathcal{J}$ on $\mathcal{A}^*$ on the $\mathcal{J}$-invariant spaces $\Gamma_{\mathfrak{a}} \cap T_{\mathfrak{a}}(\mathcal{A}^{\mathrm{HE}})$.

Generalized symplectic reduction. Let K be a Lie group, $\pi : P \to B$ a principal K-bundle with a connection A and let η be a closed K-invariant 2-form on P. We will *not* assume that η is non-degenerate. Let m be a *formal moment map* for η, i.e. a K-equivariant map $m : P \to \mathfrak{k}^\vee$ satisfying the usual condition

$$\langle dm, \xi \rangle = -\iota_{\xi\#} \eta \ \forall \xi \in \mathfrak{k} \ . \tag{73}$$

The sign on the right takes into account that K acts one on P from the right. One can project η on the base B using restriction to the A-horizontal spaces, but the obtained 2-from η_B will not be closed in general.

Let F_A be the curvature 2-form of A. We define the 2-form $\langle m, F_A \rangle$ on the base B by

$$\langle m, F_A \rangle(a, b) := \langle m(p), F_{A,p}(\tilde{a}, \tilde{b}) \rangle \ , \tag{74}$$

where a, $b \in T_x(B)$, $\pi(p) = x$ and $\tilde{a}$, $\tilde{b}$ are horizontal lifts of a and b at p.

Proposition 5.14 *It holds*

$$d(\eta_B - 2\langle m, F_A \rangle) = 0 \ . \tag{75}$$

In particular, if 0 is a regular value of m, then the restriction of η_B to the generalized symplectic reduction $m^{-1}(0)/K$ is closed.

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Proof:

For vector fields a, b on B one has the formula

$$[\tilde{a}, \tilde{b}] = \widetilde{[a, b]} - F_A(\tilde{a}, \tilde{b})^\# \ ,$$

where $\tilde{}$ stands for horizontal lifts. Taking into account that η is closed, one gets the following formula for $d\eta_B$:

$$d\eta_B(a, b, c) = -\eta(F_A(\tilde{a}, \tilde{b})^\#, \tilde{c}) + \eta(F_A(\tilde{a}, \tilde{c})^\#, \tilde{b}) - \eta(F_A(\tilde{b}, \tilde{c})^\#, \tilde{a}) =$$

$$\langle dm(\tilde{c}), F_A(\tilde{a}, \tilde{b})\rangle - \langle dm(\tilde{b}), F_A(\tilde{a}, \tilde{c})\rangle + \langle dm(\tilde{a}), F_A(\tilde{b}, \tilde{c})\rangle = 2d(\langle m, F_A\rangle)(a, b, c)$$

by the Bianchi identity. ∎

There is a simple way to endow P with a pair (η, m) consisting of a K-invariant closed 2-form and a formal moment map m for η: choosing a section in a bundle associated with a Hamiltonian K-action on a a symplectic fibre F:

Remark 5.15 *Let (F, ω_F) be a symplectic manifold endowed with a left symplectic K-action, μ a moment map for this action and $f : P \to F$ a K-equivariant map. Then $\eta := f^*(\omega_F)$ is closed and K-invariant, and $m := \mu \circ f$ is a formal moment map for η.*

Note that the equivariance property of f implies the following transformation rule for the fundamental vector fields:

$$f_*(u_p^\#) = -u_{f(p)}^\# \ \forall p \in P \ , \ \forall u \in \mathfrak{k} \ .$$

The universal bundle, the universal connection, the universal section. We regard the associated bundle

$$\hat{\mathbb{P}} := \mathcal{A}^* \times_{\mathcal{K}} \hat{P}$$

over $\mathcal{B}^*$ as a $\hat{K}$-bundle over $\mathcal{B}^* \times X$. The map

$$\Phi : \hat{\mathbb{P}} \longrightarrow F \ , \ \Phi[\hat{A}, \varphi, p] := \varphi(p)$$

is well-defined and $\hat{K}$-equivariant, so it can be regarded as section in *the universal associated bundle*

$$\mathbb{E} := \hat{\mathbb{P}} \times_{\hat{K}} F \ .$$

Using the horizontal spaces of Γ and of the variable connection $\hat{A}$, one constructs in the same way as in section 5.1.3 a universal connection $\mathbb{A}$ on $\hat{\mathbb{P}}$.

The curvature $\mathbb{F}$ of this connection is given by formulae similar to (71)

$$\mathbb{F}_{[\hat{A},\varphi,p]}(u,v) = F_{\hat{A},p}(u,v) \ , \ \mathbb{F}_{[\hat{A},\varphi,p]}((\alpha,\psi),v) = \alpha(v) \ ,$$

$$\mathbb{F}_{[\hat{A},\varphi,p]}((\alpha,\psi),(\beta,\chi)) = F_\Gamma((\alpha,\psi),(\beta,\chi))$$

for tangent vectors $u, v \in T_x(X)$ and horizontal vectors (α,ψ), $(\beta,\chi) \in \Gamma_{(\hat{A},\varphi)}$.

Remark 5.16 *The restriction of the universal pair $(\mathbb{A},\Phi)$ to $[\mathcal{M}^{\mathrm{HE}}]^*_{\mathrm{reg}} \times X$ is an integrable pair.*

For a $(2n-2)$-form ν on X we get as in the previous section

$$[p_{\mathcal{B}^*}]_*[h(\mathbb{F},\mathbb{F}) \wedge p_X^*(\nu)](a,b) =$$

$$= \int_X \left[-h(\alpha \wedge \beta) + 2h\left(\mathrm{pr}_{\mathfrak{k}}(F_{\hat{A}}), F_\Gamma((\alpha,\psi),(\beta,\chi))\right) \right] \wedge \nu \ , \qquad (76)$$

where $a, b \in T_{[\hat{A},\varphi]}(\mathcal{B}^*)$ and (α,ψ), (β,χ) are Γ-horizontal lifts of these vectors at $(\hat{A},\varphi)$. Note that, for $\nu = \omega_g^{n-1}$, the first term on the right coincides with the first term of $\Omega(a,b)$.

Consider now the $\hat{K}$-invariant closed form $\eta := \Phi^*(\omega_F)$ on $\hat{\mathbb{P}}$. This form admits a formal moment map in the sense of the previous paragraph, namely $m := \mu \circ \Phi$ (see Remark 5.15).

The projection of η on the basis $\mathcal{B}^* \times X$ is given by

$$\eta_{\mathcal{B}^* \times X}((a,u),(b,v)) = \omega_F(\psi(x),\chi(x)) \ ,$$

where $u, v \in T_x(X)$, and $\psi, \chi \in \varphi^*(V)$ are the second components of the Γ-horizontal lifts (α,ψ), (β,χ) of a and b in $(\hat{A},\varphi)$.

Applying formula (75) and taking into account that h is negative definite on $\mathfrak{k}$, we obtain

$$d[\eta_{\mathcal{B}^* \times X} + 2h(m,\mathbb{F})] = 0 \ .$$

In particular, for every $2n$-form σ on X, the form obtained by fibre integration along the X fibres

$$[p_{\mathcal{B}^*}]_* \left[(\eta_{\mathcal{B}^* \times X} + 2h(m,\mathbb{F})) \wedge p_X^*(\sigma) \right]$$

will be closed on $\mathcal{B}^*$. But obviously $h(m,\mathbb{F}) \wedge p_X^*(\sigma) = h(m,F_\Gamma) \wedge p_X^*(\sigma)$. Therefore, we obtain

Lemma 5.17 *For every $\sigma \in A^{2n}(X)$ the form Ω_σ on $\mathcal{B}^*$ defined by*

$$\Omega_\sigma(a, b) = \int_X [\omega_F(\psi, \chi) + 2h(\mu(\varphi), F_\Gamma((\alpha, \psi), (\beta, \chi)))] \, \sigma \qquad (77)$$

is closed on $\mathcal{B}^*$.

Note that the first term of $(n-1)!\Omega_{vol_g}(a, b)$ is just the second term of $\Omega(a, b)$ (which was missing in (76)).

Specializing (76) and (77) to the case $\sigma = vol_g = \frac{1}{n!}\omega_g^n$, $\nu = \omega_g^{n-1}$ and taking the sum of these formulae, we get

$$\left\{ [p_{\mathcal{B}^*}]_* [h(\mathbb{F}, \mathbb{F}) \wedge p_X^*(\nu)] + (n-1)!\Omega_{vol_g} \right\} (a, b) =$$

$$= \Omega(a, b) + \int_X 2h \left(\omega_g^{n-1} \wedge \mathrm{pr}_{\mathfrak{k}}(F_{\hat{A}}) + (n-1)!vol_g \, \mu(\varphi) \, , \, F_\Gamma((\alpha, \psi), (\beta, \chi)) \right) =$$

$$= \Omega(a, b) + (n-1)! \int_X 2h \left(\Lambda_g \mathrm{pr}_{\mathfrak{k}}(F_{\hat{A}}) + \mu(\varphi) \, , \, F_\Gamma((\alpha, \psi), (\beta, \chi)) \right) vol_g \qquad (78)$$

Restricting to the moduli space, one gets

$$[p_{\mathcal{B}^*}]_* [h(\mathbb{F}, \mathbb{F}) \wedge p_X^*(\omega_g^{n-1})] \Big|_{[\mathcal{M}^{\mathrm{HE}}]^*_{\mathrm{reg}}} + (n-1)!\Omega_{vol_g} \Big|_{[\mathcal{M}^{\mathrm{HE}}]^*_{\mathrm{reg}}} = \Omega \; . \qquad (79)$$

Since the characteristic form $h(\mathbb{F}, \mathbb{F})$ is closed of pure type (2,2) (see Remark 5.16) and Ω_{vol_g} is closed by Lemma 5.17, this shows

Theorem 5.18 *The $(1, 1)$-form Ω associated with the canonical Hermitian metric H on the smooth part $[\mathcal{M}_g^{\mathrm{st}}]_{\mathrm{reg}}$ of the moduli space $\mathcal{M}_g^{\mathrm{st}}$ has the following property:*

1. *it is Kähler, when $d(\omega_g^{n-1}) = 0$ (i.e. when g is semi-Kähler),*

2. *it satisfies the identity $\partial\bar{\partial}\Omega = 0$, when g is Gauduchon.*

Remark 5.19 *We believe that, for a given compact complex manifold Y of dimension $d \geq 3$, the existence of Hermitian metric h satisfying $\partial\bar{\partial}\omega_h = 0$ is a very restrictive condition. Therefore, smooth compact complex subspaces of any moduli space $\mathcal{M}_g^{\mathrm{st}}$ of oriented holomorphic pairs (or oriented holomorphic bundles) have very special differential geometric properties.*

5.2.3 Application: Canonical metrics on Douady Quot spaces

Let $\mathcal{E}_0$ be a fixed holomorphic vector bundle of rank r_0 on X and E a differentiable rank r vector bundle. Denote by $Quot_{\mathcal{E}_0}^E$ the Quot space of quotients $q : \mathcal{E}_0 \to \mathcal{Q}$ of $\mathcal{E}_0$ with locally free kernel of differentiable type E. This quot space, which is an open subspace of the Douady space $Quot_{\mathcal{E}_0}$ of all quotients of $\mathcal{E}_0$, has a gauge theoretical description [OT3]: it can be identified with the moduli space of pairs $(\mathcal{E}, \varphi)$, where $\mathcal{E}$ is a holomorphic structure on E and $\varphi : \mathcal{E} \to \mathcal{E}_0$ is a holomorphic *sheaf monomorphism*, modulo the gauge group $\mathcal{G} := \Gamma(X, GL(E))$.

However, it is not clear at all whether the condition "φ is a sheaf monomorphism" is a stability condition.

Using our general formalism explained in the introduction, the moduli problem for holomorphic pairs $(\mathcal{E}, \varphi)$ with $\varphi : \mathcal{E} \to \mathcal{E}_0$ corresponds to the system

$$\hat{G} = GL(r, \mathbb{C}) \times GL(r_0, \mathbb{C}), \ G = GL(r, \mathbb{C}), G_0 = GL(r_0, \mathbb{C}) \ , F = \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^{r_0}) \ .$$

A moment map for the $K = U(r)$-action on F has the form

$$\mu_t(f) = \frac{i}{2} f^* \circ f - it\mathrm{id}, \ t \in \mathbb{R} \ .$$

Using the explicit form of the maximal weight function associated with a linear representation given in section 1.3.1, one obtains the following stability condition corresponding to μ_t. Put $\tau := \frac{(n-1)! Vol_g(X)}{2\pi} t$.

Definition 5.20 *A holomorphic pair $(\mathcal{E}, \varphi)$ is called τ-stable if for every nontrivial subsheaf $\mathcal{F}$ of $\mathcal{E}$, the following holds:*

1. *$\mu_g(\mathcal{E}/\mathcal{F}) > -\tau$ if $\mathrm{rk}(\mathcal{F}) < r$.*

2. *$\mu_g(\mathcal{F}) < -\tau$ if $\mathcal{F} \subset \ker(\varphi)$.*

Here μ_g denotes the g-slope $\deg_g / \mathrm{rk}$. Note that the g-degree $\deg_g$ is *not* a topological invariant for general Gauduchon metrics g (see [LT]).

Our next purpose is to compare the Quot space $Quot_{\mathcal{E}_0}^E$ with a moduli space of τ-stable pairs. The result is

Theorem 5.21 *Fix E and $\mathcal{E}_0$ as above.*

1. *Suppose that g is semi-Kähler. Then for sufficiently large $\tau \in \mathbb{R}$, the following are equivalent:*

   (a) *$(\mathcal{E}, \varphi)$ is τ-stable.*

(b) φ is injective.

2. *Suppose that g is Gauduchon. Then for any $s \in \mathbb{R}$ there exists $\tau(d, \mathcal{E}_0) \in \mathbb{R}$ such that for every holomorphic pair $(\mathcal{E}, \varphi)$, $\varphi : \mathcal{E} \to \mathcal{E}_0$ with $\deg_g(\mathcal{E}) \geq s$ and every $\tau \geq \tau(s, \mathcal{E}_0)$ the following are equivalent:*

   (a) $(\mathcal{E}, \varphi)$ is τ-stable.
   (b) φ is injective.

Proof: For the semi-Kählerian case the proof can be found in [OT3]. The second statement follows using the same methods, but taking into account that the g-degree is no longer a topological invariant in the Gauduchon case. ∎

Let
$$d_g : Quot_{\mathcal{E}_0}^E \longrightarrow \mathbb{R}$$
be the real analytic map defined by $d_g : (\mathcal{E}, \varphi) := \deg_g(\mathcal{E})$. The second statement of the theorem above shows that the open subspaces
$$[Quot_{\mathcal{E}_0}^E]_s := d_g^{-1}(-s, \infty)$$
of $Quot_{\mathcal{E}_0}^E$ can be identified with moduli spaces of stable oriented pairs. By Theorem 5.18, we obtain

Corollary 5.22 1. *Suppose that g is semi-Kähler. Then the smooth part of the quot space $Quot_{\mathcal{E}_0}^E$ admits a canonical Kähler metric.*

2. *Suppose that g is Gauduchon. Then, for any $s \in \mathbb{R}$, the smooth part of the truncated quot space $[Quot_{\mathcal{E}_0}^E]_s$ admits a canonical Hermitian metric whose Kähler form Ω satisfies the identity $\partial\bar{\partial}\Omega = 0$.*

This result can be specialized to Douady spaces spaces of effective divisors. For any class $m \in \mathbb{H}^2(X, \mathbb{Z})$, let $\mathcal{D}ou(m)$ be the Douady space of effective divisors in X representing the homology class $PD(m)$. *On Gauduchon manifolds, these spaces are in general not compact.* $\mathcal{D}ou(m)$ can be identified with the quot space $Quot_{\mathcal{O}_X}^{M^\vee}$, where M is a differentiable line bundle with Chern class m.

For every $s \in \mathbb{R}$, we define the *truncated* Douady space $\mathcal{D}ou(m)_s$ by
$$\mathcal{D}ou(m)_s := \{D \in \mathcal{D}ou(m) \mid Vol_g(D) = \deg_g(\mathcal{O}(D)) < s\}.$$
These spaces are relatively compact in $\mathcal{D}ou(m)$, by Bishop's compactness theorem. One has
$$\mathcal{D}ou(m)_s = [Quot_{\mathcal{O}_X}^{M^\vee}]_s \ ,$$
so our result gives a canonical Kählerian metric on every $\mathcal{D}ou(m)$ when g is semi-Kähler, and a canonical metric satisfying $\partial\bar{\partial}\Omega = 0$ on every truncated space $\mathcal{D}ou(m)_s$.

Corollary 5.23 *On a Gauduchon surface X with odd $b_1(X)$, the moduli space of irreducible solutions of a (suitably perturbed) abelian Seiberg-Witten equations is a disjoint union of two truncated Douady spaces of effective divisors (see [OT7]). Therefore, one gets canonical Hermitian metrics with the mentioned property on all these moduli spaces of irreducible abelian monopoles.*

Remark 5.24 *It might be interesting to study the behavior of the obtained metrics on $[Quot_{\mathcal{E}_0}^E]_s$ as $s \to \infty$.*

5.3   Non-abelian monopoles on Gauduchon surfaces

5.3.1   Oriented rank 2 holomorphic pairs

Let E be a rank 2 complex vector bundle over a compact Gauduchon manifold (X, g), and let $\mathcal{L}$ be a fixed holomorphic structure on its determinant line bundle $L := \det(E)$.

We are interested in the classification of *oriented holomorphic pairs of type* $(E, \mathcal{L})$, i.e. of pairs $(\mathcal{E}, \varphi)$ consisting of a holomorphic structure $\mathcal{E}$ on E inducing $\mathcal{L}$ on L and a holomorphic section $\varphi \in H^0(X, \mathcal{E})$. This complex geometric moduli problem corresponds – in our general setting – to the system

$$G = SL(2) \ , \ \hat{G} = GL(2) \ , \ G_0 = \mathbb{C}^* \ , \ F = \mathbb{C}^2 \ .$$

Remark 5.25 *Non-oriented pairs have been considered by many authors (see e.g. [Bra], [Th], [HL]). Oriented holomorphic pairs have been first introduced in [OT2] and [Te2] in the Kählerian framework in order to give a complex geometric interpretation for the moduli spaces of $PU(2)$-monopoles.*

The stability condition for oriented rank 2 holomorphic pairs is quite simple:

Definition 5.26 *An oriented pair $(\mathcal{E}, \varphi)$ of type $(E, \mathcal{L})$ is called*

1. *stable if one of the following conditions holds:*

   (a) *$\varphi = 0$ and $\mathcal{E}$ is a g-stable holomorphic bundle;*
   (b) *$\varphi \neq 0$ and $\mu_g(\mathcal{O}_X(D_\varphi)) < \mu_g(\mathcal{E}) = \frac{1}{2}\deg_g(\mathcal{L})$, where D_φ is the divisorial component of the vanishing locus $Z(\varphi)$;*

2. *g-polystable if it is g-stable, or $\varphi = 0$ and $\mathcal{E}$ is a g-polystable holomorphic bundle.*

Note that, contrary to the case of non-oriented pairs, this stability condition is not parameter dependent (compare to the previous section in this article and with [Bra]).

Remark 5.27 *The stability condition for higher rank oriented pairs is much more difficult (see [OST] for the algebraic case).*

We will denote by $\mathcal{M}^{\mathrm{st}}(E,\mathcal{L})$ the moduli space of stable oriented pairs of type $(E,\mathcal{L})$. The stabilizer of a stable oriented pair of the form $(\mathcal{E},0)$ with respect to the natural action of the complex gauge group $\mathcal{G} := \Gamma(X, SL(E))$ is $\mathbb{Z}_2$, so these moduli spaces have at least $\mathbb{Z}_2$-orbifold singularities along the locus $\varphi = 0$.

Moduli spaces of oriented rank 2 holomorphic pairs are very interesting geometric objects for the following reason:

Remark 5.28 *The moduli space $\mathcal{M}^{\mathrm{st}}(E,\mathcal{L})$ comes with a natural $\mathbb{C}^*$-action, given by $(\zeta, (\mathcal{E}, \varphi)) \mapsto (\mathcal{E}, \zeta\varphi)$. In the terminology of [OST], [OT6], $\mathcal{M}^{\mathrm{st}}(E,\mathcal{L})$ is a "master space" associated to the coupling of holomorphic bundles with holomorphic sections.*

The fixed point locus of this $\mathbb{C}^*$-action is described as follows:

Lemma 5.29 *Suppose that (X, g) is a Gauduchon surface, and denote*

$$l := c_1(E) \ , \ \ \lambda := \frac{1}{2}\deg_g(\mathcal{L}) \ , \ \ c := c_2(E) \ .$$

Then the fixed point locus of the $\mathbb{C}^*$-action on $\mathcal{M}^{\mathrm{st}}(E,\mathcal{L})$ decomposes as

$$\mathcal{M}^{\mathrm{st}}(E,\mathcal{L})^{\mathbb{C}^*} = \mathcal{M}^{\mathrm{st}}_{\mathcal{L}}(E) \coprod \left[\coprod_{m(l-m)=c} \mathcal{D}ou(m)_\lambda \right] \ ,$$

where $\mathcal{M}^{\mathrm{st}}_{\mathcal{L}}(E)$ denotes the moduli space of g-stable holomorphic structures on E inducing $\mathcal{L}$ on L.

The truncated Douady spaces $\mathcal{D}ou(m)_\lambda$ have been defined in the previous section. On higher dimensional manifolds one must take the union of $\mathcal{D}ou(m)_\lambda$, over all classes m such that the line bundle M of Chern class m is a topological summand of E.

Proof: The part $\mathcal{M}^{\mathrm{st}}_{\mathcal{L}}(E)$ is the locus $\varphi = 0$, which is obviously part of the fixed point locus. A truncated Douady space $\mathcal{D}ou(m)_\lambda$ with $m(l-m) = c$ is embedded in $\mathcal{M}^{\mathrm{st}}(E,\mathcal{L})^{\mathbb{C}^*}$ as follows: To every $D \in \mathcal{D}ou(m)_\lambda$ we associate the stable pair $(\mathcal{O}_X(D) \oplus [\mathcal{L} \otimes \mathcal{O}_X(-D)], \varphi_D)$, where φ_D is the canonical section of $\mathcal{O}_X(D)$. It is easy to see that any fixed point of the $\mathbb{C}^*$-action has one of these two types. ∎

Remark 5.30 *When X is a surface, the space $\mathcal{M}^{\mathrm{st}}_{\mathcal{L}}(E)$ can be identified – via the Kobayashi-Hitchin isomorphism – with a Donaldson moduli space of oriented, projectively ASD $U(2)$-connections. When the base is simply connected, this space can be further identified with a moduli space of $PU(2)$-instantons.*

The spaces $\mathcal{D}ou(m)_\lambda$ are related to Seiberg-Witten moduli spaces, as pointed out in Corollary 5.23. We will see that the total space $\mathcal{M}^{\mathrm{st}}(E,\mathcal{L})$ is part of a moduli space of non-abelian monopoles.

On surfaces, the moduli spaces $\mathcal{M}^{\mathrm{st}}(E,\mathcal{L})$ have natural compactifications, which are complex schemes when the base surface is algebraic.

Any non-constant $\mathbb{C}^*$-orbit in $\mathcal{M}^{\mathrm{st}}(E,\mathcal{L})$ connects either a point of Donaldson type to a point of Seiberg-Witten type, or two points of Seiberg-Witten type (one of which can belong to the compactification of $\mathcal{M}^{\mathrm{st}}(E,\mathcal{L})$).

The complex geometric moduli problem for oriented holomorphic pairs has the following differential geometric analogue.

Fix a Hermitian metric h on E and let A_0 be the Chern connection of the pair $(\mathcal{L}, \det(h))$. Our configuration space is the space $\mathcal{A} := \mathcal{A}_{A_0}(E) \times A^0(E)$, where $\mathcal{A}_{A_0}(E)$ is the space of unitary connections on E which induce A_0 on L. The moduli space of *oriented vortices* is defined by

$$\mathcal{M}^{\mathrm{HE}}(E, A_0) := {}^{\mathcal{A}^{\mathrm{HE}}}\!/_{\mathcal{K}} \ ,$$

where $\mathcal{A}^{\mathrm{HE}}$ is the space of integrable solutions of the *projective vortex equation* (see [Te2] for the Kählerian case):

$$i\Lambda_g F_A^0 + \frac{1}{2}(\varphi\bar{\varphi})_0 = 0 \ ,$$

and $\mathcal{K}$ is the gauge group $\Gamma(X, SU(E))$. Here F_A^0 denotes the trace-free part of the curvature, which can be identified with the curvature of the associated $PU(2)$-connection. A pair (A, φ) will be called irreducible if its infinitesimal stabilizer vanishes. The stabilizer of an irreducible pair is either trivial or $\mathbb{Z}_2$.

In our case, the Kobayashi-Hitchin isomorphism gives:

Theorem 5.31 *Let (X, g) be a Gauduchon manifold, (E, h) a Hermitian rank 2 bundle on X, $\mathcal{L}$ a holomorphic structure on $L := \det(E)$, and A_0 the Chern connection of the pair $(\mathcal{L}, \det(h))$. The map $(A, \varphi) \mapsto (\bar{\partial}_A, \varphi)$ induces a real analytic isomorphism of moduli spaces*

$$[\mathcal{M}^{\mathrm{HE}}(E, A_0)]^* \xrightarrow{\ KH\ } \mathcal{M}^{\mathrm{st}}(E, \mathcal{L}) \ .$$

This isomorphism extends to a homeomorphism $\mathcal{M}^{\mathrm{HE}}(E, A_0) \simeq \mathcal{M}^{\mathrm{pst}}(E, \mathcal{L})$.

Remark 5.32 *1. With an appropriate definition of (poly)stability for oriented pairs (see [OST] for the algebraic and [OT8] for the Kählerian case), the theorem holds for arbitrary rank r.*

2. On surfaces, the spaces $\mathcal{M}^{\mathrm{HE}}(E, A_0)$ have Uhlenbeck-type compactifications, which are complex schemes when the base is algebraic.

5.3.2 Complex geometric description of non-abelian monopoles

It is known (see [FL1], [FL2], [OT2] [PT], [Te5]) that the most natural way to prove the Witten conjecture relating the Donaldson invariants to the Seiberg-Witten invariants is to consider certain moduli spaces of non-abelian monopoles, which contain both Donaldson moduli spaces and moduli spaces of abelian monopoles. Substantial progress towards a complete proof of the Witten conjecture in full generality (for arbitrary simple type manifolds) was obtained in [FL2]. The relation between the two type of differential topological type of invariants has more than theoretical importance: it plays an important role in the recent spectacular results of Kronheimer-Mrowka in 3-dimensional topology [KM].

On the other hand, the moduli spaces of non-abelian monopoles are interesting, difficult and mysterious objects, so we believe that, independently of the Witten conjecture, it is important and interesting to have methods for describing such moduli spaces explicitly. The Kobayashi-Hitchin correspondence yields such a method even on Gauduchon surfaces.

Let M be a closed oriented Riemannian 4-manifold and τ a $Spin^c$-structure on M. Denote by $\Sigma^{\pm}$, $\Sigma := \Sigma^+ \oplus \Sigma^-$ the spinor bundles of τ, by $D := \det(\Sigma^{\pm})$ its determinant line bundle and by $\gamma : \Lambda^1_M \to \mathrm{End}(\Sigma)$ the Clifford map.

Fix a Hermitian 2-bundle E of determinant L on M. We choose parameter connections A_0 and B_0 on L and D.

The configuration space for the non-abelian monopole equations is

$$\mathcal{A} := \mathcal{A}_{A_0}(E) \times A^0(\Sigma^+ \otimes E) \ ,$$

and the gauge group is $\mathcal{K} := \Gamma(M, SU(E))$. Fix a complex form $\beta \in A^1(M, \mathbb{C})$. The monopole equations associated with the data (τ, B_0, β, A_0) read

$$\begin{cases} \displaystyle{\not{D}}_A \Psi + \gamma(\beta) &=& 0 \\ \gamma(F_A^0) &=& (\Psi \otimes \bar{\Psi})_0 \end{cases} \tag{80}$$

where the term $(\Psi \otimes \bar{\Psi})_0$ stands for the projection of

$$\Psi \otimes \bar{\Psi} \in A^0(\mathrm{End}(\Sigma) \otimes \mathrm{End}(E))$$

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on $A^0(\mathrm{End}_0(\Sigma) \otimes \mathrm{End}_0(E))$, and $\displaystyle{\not}D_A$ is the coupled Dirac operator associated with the connection A and the abelian connection B_0 on D. It is easy to see that this projection belongs to

$$\mathrm{Herm}_0(\Sigma^+) \otimes \mathrm{Herm}_0(E) \xrightarrow{\ \simeq \gamma^{-1}\ } \Lambda_+^2 \otimes su(E) \ .$$

Standards methods ([FL1], [Te5]) show that the moduli space

$$\mathcal{M} := {}^{\mathcal{A}^{SW}}\!/_{\mathcal{K}}$$

of solutions of this equation is a finite dimensional real analytic space which has a natural Uhlenbeck-type compactification. This is constructed in the same way as for the moduli spaces of instantons in Donaldson theory, by adding a *finite* number of strata of the form $S^k(X) \times \mathcal{M}_k$, where $\mathcal{M}_k$ are Seiberg-Witten moduli spaces associated with bundles E_k having $\det(E_k) = \det(E)$ and $c_2(E_k) = c_2(E) - k$, $k \geq 1$ (see [FL1], [Te5]).

The moduli space $\mathcal{M}$ comes with a natural S^1-action (given by scalar multiplication of the spinor component) and one can easily see that the fixed point locus $\mathcal{M}^{S^1}$ decomposes as a union of the Donaldson moduli space $\mathcal{D} := \mathcal{M}_{A_0}^{\mathrm{ASD}}(E)$ of oriented projectively ASD connections (see section 5.1.2) and the union $\mathcal{SW}$ of a finite number of Seiberg-Witten moduli spaces of abelian monopoles. The fundamental idea of the "cobordism strategy" ([FL1], [OT2] [PT], [Te5]) is to remove a small neighborhood U of $\mathcal{M}^{S^1}$ in $\mathcal{M}$ and to use the quotient ${}^{[\mathcal{M} \setminus U]}\!/_{S^1}$ as a "homology equivalence" between $\mathcal{D}$ and $\mathcal{SW}$. The problem is that new Seiberg Witten moduli spaces can appear in the lower strata of the Uhlenbeck compactification, and the structure of this compactification is very complicated [FL1], [FL2].

Let now (X, g) be a Gauduchon surface and τ_{can} the canonical $Spin^c$ structure of X. The spinor bundles of τ_{can} are:

$$\Sigma_{\mathrm{can}}^+ = \Lambda^{0,0} \oplus \Lambda^{0,2} \ , \ \Sigma_{\mathrm{can}}^- = \Lambda^{0,1} \ ,$$

so that the determinant line bundle D_{can} is $K_X^\vee = \wedge^2(T_X^{1,0})$. Denote by B_{can} the connection on D_{can} induced by the Chern connection of the pair $(\mathcal{T}_X, g)$.

A spinor $\Psi \in \Sigma_{\mathrm{can}}^+ \otimes E$ decomposes as $\Psi = \varphi + \alpha$, where $\varphi \in A^0(E)$ and $\alpha \in A^{0,2}(E) = A^0(E \otimes K_X^\vee)$.

We will suppose that the Chern class $c_1(E)$ belongs to the Neron-Severi group $NS(X)$ of X, and that the parameter connection A_0 is integrable.

Our perturbation 1-form will be

$$\beta_{\mathrm{can}} := -\frac{1}{4}\theta_g \ ,$$

where θ_g is the torsion form of g, defined by the equality $d\omega_g = \omega_g \wedge \theta_g$. This particular choice is very important: the corresponding perturbed coupled Dirac operator $\not{D}_A + \gamma(\beta_{\mathrm{can}})$ coincides with the "Dolbeault-Dirac"-operator

$$\mathcal{D}_A := \sqrt{2}(\bar{\partial}_A + \bar{\partial}_A^*) \ .$$

([Bi]). As in the Kählerian case [Te2], one can prove the following *decoupling theorem*, which extends to the non-abelian Seiberg-Witten theory the well-known Witten decoupling theorem for abelian monopoles on Kählerian surfaces [W]:

Theorem 5.33 *A pair*

$$(A, \varphi + \alpha) \in \mathcal{A}_{A_0}(E) \times \left[A^0(E) \oplus A^{0,2}(E) \right]$$

solves the monopole equations associated with the data $(\tau_{\mathrm{can}}, B_{\mathrm{can}}, \beta_{\mathrm{can}}, A_0)$ *if and only if A is integrable and one of the following holds:*

1. *$\alpha = 0$, $\bar{\partial}_A \varphi = 0$ and $i\Lambda_g F_A^0 + \frac{1}{2}(\varphi\bar{\varphi})_0 = 0$,*

2. *$\varphi = 0$, $\partial_A \alpha = 0$ and $i\Lambda_g F_A^0 - \frac{1}{2} * (\alpha \wedge \bar{\alpha})_0 = 0$.*

This theorem shows that the moduli space $\mathcal{M}$ of solutions of the non-abelian Seiberg-Witten equations associated with the data $(\tau_{\mathrm{can}}, B_{\mathrm{can}}, \beta_{\mathrm{can}}, A_0)$ decomposes as union of two closed subspaces $\mathcal{M}_I$, $\mathcal{M}_{II}$ intersecting along the Donaldson moduli space $\mathcal{M}_{A_0}(E)$ of oriented, projectively ASD connections on E.

The first part $\mathcal{M}_I$ is just the moduli space $\mathcal{M}^{\mathrm{HE}}(E, A_0)$ of oriented vortices, so it has a complex geometric interpretation by Theorem 5.31. The second part $\mathcal{M}_{II}$ can be identified with $\mathcal{M}^{\mathrm{HE}}(E^\vee \otimes K_X, A_0^\vee \otimes [B_{\mathrm{can}}^\vee]^{\otimes 2})$ via the map

$$(A, \alpha) \mapsto (A^\vee \otimes B_{\mathrm{can}}^\vee, \bar{\alpha}) \ .$$

Therefore, we get the following complex geometric description of the moduli space of non-abelian monopoles associated with the considered data:

Theorem 5.34 *The moduli space $\mathcal{M}$ of solutions of the monopole equations associated with the data $(\tau_{\mathrm{can}}, B_{\mathrm{can}}, \beta_{\mathrm{can}}, A_0)$ decomposes as a union of two closed subspaces*

$$\mathcal{M} = \mathcal{M}_I \cup \mathcal{M}_{II} \ .$$

These spaces can be identified with moduli spaces of polystable rank 2 oriented holomorphic pairs as follows:

$$\mathcal{M}_I \simeq \mathcal{M}_{\mathcal{L}}^{\mathrm{pst}}(E) \ , \ \ \mathcal{M}_{II} \simeq \mathcal{M}_{\mathcal{L}^\vee \otimes \mathcal{K}_X^2}^{\mathrm{pst}}(E^\vee \otimes K_X) \ .$$

The intersection $\mathcal{M}_I \cap \mathcal{M}_{II}$ is a Donaldson moduli space of oriented projectively ASD unitary connections, and can be identified with the moduli space of polystable oriented holomorphic structures $\mathcal{M}_{\mathcal{L}}^{\mathrm{pst}}(E)$.

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Remark 5.35 *Using Theorem 5.34 one can describe explicitly moduli spaces of non-abelian monopoles as S^1-spaces. An explicit example in the algebraic case – which illustrates very clearly the cobordism strategy – is given in [Te2].*

In the non-algebraic framework, one has to deal with a substantial difficulty: the appearance of non-filtrable bundles (i.e. of bundles which admit non proper subsheaf of lower rank). For such bundles there exist no general construction and classification methods.

Note however that this difficulty concerns only the "Donaldson part" of the moduli space (the intersection $\mathcal{M}_I \cap \mathcal{M}_{II}$) because, for an oriented holomorphic pair $(\mathcal{E}, \varphi)$ with $\varphi \neq 0$, the bundle $\mathcal{E}$ will be automatically filtrable.

6 Appendix

6.1 Chern connections

Let G be a complex Lie group and Q a principal G-bundle over a complex manifold M. By definition, an (almost) holomorphic structure in Q is an (almost) complex structure in the total space of Q which makes the projection on M (almost) holomorphic and the action $G \times Q \to Q$ holomorphic. In other words, an (almost) holomorphic structure in the principal bundle Q is a G-invariant (almost) complex structure on its total space which agrees with the complex structure induced by the complex structure in the Lie algebra $\mathfrak{g}$ on the vertical subbundle $V_Q \subset T_Q$, and with the complex structure in M on the quotient bundle T_Q/V_Q.

A holomorphic structure on Q can be alternatively defined as the data of a maximal system of trivializations with holomorphic transition functions.

If K is a maximal compact subgroup of a reductive group G and $P \subset Q$ is a K-reduction of Q, then the *Chern correspondence* identifies the space $\bar{\mathcal{A}}(Q)$ of almost holomorphic structures on the bundle Q with the space $\mathcal{A}(P))$ of connections on P.

The correspondence is defined as follows: if J is an almost complex structure on the G-bundle Q, then the horizontal spaces of the associated connection $A \in \mathcal{A}(P)$ are given by:

$$T_p^{\mathrm{horiz}}(P) = T_p(P) \cap J(T_p(P)) \ , \ p \in P \ .$$

The tangent space $T_p(Q)$ decomposes as

$$T_p(Q) = T_p(P) \oplus T_p(P)^\perp = T_p^{\mathrm{horiz}}(P) \oplus T_p^{\mathrm{vert}}(P) \oplus T_p(P)^\perp \ ,$$

where $T_p(P)^\perp = \{a_p^\#| \ a \in i\mathfrak{k}\}$.

The Chern correspondence identifies the space $\mathcal{H}(Q)$ of holomorphic structures on Q with the space

$$\mathcal{A}^{1,1}(P) := \{A \in \mathcal{A}(P)| \ F_A \in \mathcal{A}^{1,1}(\mathrm{ad}(P))\}$$

of *integrable* connections on P.

We will denote by J^A the (almost) holomorphic structure corresponding to the connection A and by A_J (or, more precisely $A_{P,J}$) the connection corresponding to J.

The aim of this section is to study the dependence of this connection and its curvature on P, when J is fixed.

Proposition 6.1 *The vertical component of a vector $w \in T_p(P)$ with respect to the Chern connection $A_{P,J}$ is given by the formula*

$$\mathrm{vert}(w) = -J \left[\mathrm{pr}_{T_p(P)^\perp}(J(w))\right] \ .$$

Proof: One has

$$J(T_p(P)^\perp) = T_p^{\mathrm{vert}}(P) \ , \ \ J(T_p^{\mathrm{horiz}}(P)) = T_p^{\mathrm{horiz}}(P) \ ,$$

hence

$$w = \mathrm{horiz}(w) + \mathrm{vert}(w) \ ; \ \ J(w) = J\mathrm{horiz}(w) + J\mathrm{vert}(w) \ ,$$

so that

$$J\mathrm{vert}(w) = \mathrm{pr}_{T_p(P)^\perp}(J(w)) \ ; \ \ \mathrm{vert}(w) = -J \left[\mathrm{pr}_{T_p(P)^\perp}(J(w))\right] \ .$$

∎

Let P^0 be a fixed K-reduction of Q, and let A^0 be the Chern connection of the pair (P^0, J). One can write

$$P = e^{-\frac{s}{2}}(P^0) \ , \tag{81}$$

where $s \in A^0(P^0 \times_{\mathrm{ad}} i\mathfrak{k})$.

Proposition 6.2 *Denote by A^s the Chern connection of the pair (P, J). Regarding both connections A^s and the initial connection $A^0 := A_{P^0,J}$ as connections in Q, the relation between these two connections is*

$$A^s = A^0 + h^{-1}\partial_0 h \ ,$$

where $h := e^s$, and the term $h^{-1}\partial_0 h$ on the right is defined in the following way: regard h as a G-equivariant map $\chi : Q \to G$, and define the ad*-tensorial $(1,0)$ form $\chi^{-1}\partial_0\chi$ on Q by setting*

$$\chi^{-1}\partial_0\chi(v) = \chi^{-1}\partial\chi(\mathrm{horiz}_{A^0}(v)) = [L_{\chi(q)^{-1}}]_*\partial\chi(\mathrm{horiz}_{A^0}(v))$$

for all $q \in Q, v \in T_q(Q)$. This tensorial form can be identified with a form in $h^{-1}\partial_0 h \in A^{1,0}(X, \mathrm{ad}(Q))$.

Proof: Put $k := e^{-\frac{s}{2}}$. Choose $p \in P^0$, and let $q := k(p) = p\kappa(p) \in P^s$. We want to compute the difference $\alpha := \theta_{A^s} - \theta_{A^0}$ between the two connection forms in the point q.

Fix an A^0-horizontal vector $w \in T_p(P^0)$. The vector

$$[R_{\kappa(p)}]_*(w) \in T_q(R_{\kappa(p)}(P^0)) \subset T_q(Q)$$

will be horizontal with respect to the same connection regarded as a connection on Q.

On the other hand,

$$k_*(w) = [R_{\kappa(p)}]_*(w) + q\ \kappa(p)^{-1}d\kappa(w) \in T_q(P^s)\ .$$

Since $J(w)$ belongs to $T_p(P^0)$ too, one has

$$k_*(Jw) = [R_{\kappa(p)}]_*(Jw) + q\ \kappa(p)^{-1}d\kappa(Jw) \in T_q(P^s)\ ,$$

hence

$$[R_{\kappa(p)}]_*(Jw) \equiv -q\ \kappa(p)^{-1}d\kappa(Jw) \bmod T_q(P^s)$$

and

$$(\theta_{A^s} - \theta_{A^0})(k_*(w)) = \theta_{A^s}([R_{\kappa(p)}]_*(w) + q\ \kappa(p)^{-1}d\kappa(w)) - \kappa(p)^{-1}d\kappa(w)\ .$$

But

$$\mathrm{vert}_{A^s}([R_{\kappa(p)}]_*(w) + q\ \kappa(p)^{-1}d\kappa(w)) =$$

$$= -J\ \mathrm{pr}_{T_q(P^s)^\perp}(([R_{\kappa(p)}]_*(Jw) + Jq\ \kappa(p)^{-1}d\kappa(w)) =$$

$$= -J\ \mathrm{pr}_{T_q(P^s)^\perp}(-q\ \kappa(p)^{-1}d\kappa(Jw) + Jq\ \kappa(p)^{-1}d\kappa(w)) =$$

$$= q\{-J_{\mathfrak{g}}\ \mathrm{pr}_{\mathfrak{k}^\perp}(-\ \kappa(p)^{-1}d\kappa(Jw) +\ J_{\mathfrak{g}}(\kappa(p)^{-1}d\kappa(w))\}$$

Therefore, since $\theta_{A^s} - \theta_{A^0}$ is a tensorial form of type ad and $[R_{\kappa(p)}]_*(w) - k_*(w)$ is a vertical vector, one gets

$$\mathrm{ad}_{\kappa(p)^{-1}}(\theta_{A^s} - \theta_{A^0})(w) = (\theta_{A^s} - \theta_{A^0})([R_{\kappa(p)}]_*(w)) = (\theta_{A^s} - \theta_{A^0})(k_*(w)) =$$

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$$= -i\,\mathrm{pr}_{\mathfrak{k}^\perp}\left[-\,\kappa(p)^{-1}d\kappa(Jw) +\; i(\kappa(p)^{-1}d\kappa(w)\right] - \kappa(p)^{-1}d\kappa(w) =$$

$$= -i\,\mathrm{pr}_{i\mathfrak{k}}\left[i(i\kappa(p)^{-1}d\kappa(Jw) +\; \kappa(p)^{-1}d\kappa(w))\right] - \kappa(p)^{-1}d\kappa(w) =$$

$$= -i\,\mathrm{pr}_{i\mathfrak{k}}\left[2i\kappa(p)^{-1}\bar\partial\kappa(w)\right] - \kappa(p)^{-1}d\kappa(w) =$$

$$= -i\,\mathrm{pr}_{i\mathfrak{k}}\left[2i\kappa(p)^{-1}\bar\partial\kappa(w)\right] - \kappa(p)^{-1}d\kappa(w) =$$

$$= -i\left[i\kappa(p)^{-1}\bar\partial\kappa(w) - i\partial\kappa(w)\kappa(p)^{-1}\right] - \kappa(p)^{-1}d\kappa(w) =$$

$$= -\partial\kappa(w)\kappa(p)^{-1} - \kappa(p)^{-1}\partial\kappa(w)\ ,$$

so that

$$(\theta_{A^s} - \theta_{A^0})(w) = \mathrm{ad}_{\kappa(p)}[-\partial\kappa(w)\kappa(p)^{-1} - \kappa(p)^{-1}\partial\kappa(w)] =$$

$$[-\kappa(p)\partial\kappa(w)\kappa(p)^{-2} - \partial\kappa(w)\kappa(p)^{-1}]\ .$$

But

$$\chi^{-1}d\chi = \kappa^2\partial(\kappa^{-2}) = -\kappa(\partial\kappa)\kappa^{-2} - (\partial\kappa)\kappa^{-1}\ .$$

∎

Corollary 6.3 *For the curvature of $A^s := A_{e^{-\frac{s}{2}}(P^0),J}$ one gets the formula*

$$F_{A^s} = F_{A^0} + \bar\partial(h^{-1}\partial_0 h)\ ,$$

where $\bar\partial$ stands for the Dolbeault operator associated with the fixed holomorphic structure J on Q.

Sometimes it is more convenient to work with connections on the fixed K-principal bundle P^0. Therefore, we will also consider the connection

$$A_s := [e^{-\frac{s}{2}}]^*(A^s) \in \mathcal{A}(P^0)\ .$$

Proposition 6.4 *The connection form and the curvature form of the connection A_s are given by the formulae*

$$A_s - A^0 = k^{-1}\bar\partial_0 k - (\partial_0 k)k^{-1} = l^{-1}(\partial_0 l) - (\bar\partial_0 l)l^{-1}\ ,$$

$$F_{A_s} = \mathrm{ad}_l(F_{A^s}) = \mathrm{ad}_l(F_{A^0} + \bar\partial(h^{-1}\partial_0 h))\ ,$$

where $k := e^{-s}$ and $l := k^{-1} = \sqrt{h}$.

Proof:

$$A_s - A^0 = k^{-1}(d_{A^s}k) + h^{-1}\partial_0 h =$$

$$k^{-1}(d_0 k) + k^{-1}[-k(\partial_0 k)k^{-2} - (\partial_0 k)k^{-1}, k] - k(\partial_0 k)k^{-2} - (\partial_0 k)k^{-1} =$$

$$= k^{-1}(d_0 k) - (\partial_0 k)k^{-1} - k^{-1}(\partial_0 k) + k(\partial_0 k)k^{-2} + (\partial_0 k)k^{-1} - k(\partial_0 k)k^{-2} - (\partial_0 k)k^{-1}$$

$$= k^{-1}(\partial_0 k) + k^{-1}(\bar\partial_0 k) - (\partial_0 k)k^{-1} - k^{-1}(\partial_0 k) = k^{-1}\bar\partial_0 k - (\partial_0 k)k^{-1}\ .$$

∎

6.2 The orbits of the adjoint action. Sections in the adjoint bundle

Let K be a compact Lie group and $T \subset K$ a fixed maximal torus.

We need first to understand the structure of the quotient $\mathfrak{k}/K$, where K acts on its Lie algebra $\mathfrak{k}$ via the adjoint representation.

First of all note that any element $\varphi \in \mathfrak{k}$ is conjugate to an element in $\mathfrak{t}$. Moreover, if two elements φ_1, $\varphi_2 \in \mathfrak{t}$ are conjugate, i. e. $\mathrm{ad}_k(\varphi_1) = \varphi_2$ for some $k \in K$, then $\mathrm{ad}_k(T)$ and T are both maximal tori of K which are contained in the connected centralizer $Z_K(\varphi_2)$. Therefore, these tori are conjugate in $Z_K(\varphi_2)$, hence there exists $u \in Z_K(\varphi_2)$ such that $\mathrm{ad}_u(\mathrm{ad}_k(T)) = T$. This shows that $\kappa := uk$ belongs to the normalizer $N_K(T)$ of T in K, hence $\mathrm{ad}_\kappa(\varphi_1) = \mathrm{ad}_u(\varphi_2) = \varphi_2$, so that φ_1, φ_2 are congruent modulo $N_K(T)$. Concluding, we get an identification

$$\mathfrak{k}/K = \mathfrak{t}/N_K(T) = \mathfrak{t}/W_K(T) \ .$$

When K is connected the quotient $\mathfrak{t}/W_K(T)$ can be identified with any closed Weyl chamber $C \subset \mathfrak{t}$. Indeed, it suffices to note that two distinct elements on the boundary of C cannot be congruent modulo $W_K(T)$.

To prove this, let x, $y \in C \setminus \mathring{C}$ be two points which are congruent modulo $W_K(T)$. Let $w \in W_K(T)$ such that $w(x) = y$. Then $y \in C \cap w(C)$. Let c be the common face of C and $w(C)$ of minimal dimension which contains y.

Since the chambers C and $w(C)$ have a common face c, one can find a sequence of chambers $(C_i)_{1 \leq i \leq k}$ such that $C_1 = C$, $C_k = w(C)$ and C_i, C_{i+1} have a common codimension 1 face which contains c. It follows that C_i is mapped to C_{i+1} by the reflexion with respect to the hyperplane H_i generated by this common codimension 1 face. Therefore C is mapped to $w(C)$ by a composition of reflexions with respect to hyperplanes which contain c. Since the operation of the Weyl group is free transitive on the set of chambers, it follows that w coincides with this composition, hence it leaves y fixed.

Therefore,

Remark 6.5 *Suppose that K is connected, and let $C \subset \mathfrak{t}$ be a closed Weyl chamber. The natural map*

$$C \longrightarrow \mathfrak{k}/K$$

is a homeomorphism.

Let K be a connected compact Lie group and P a principal K-bundle over an arbitrary manifold B. We fix a closed Weyl chamber $C \subset \mathfrak{t}$. C decomposes as a disjoint union of convex sets, the open faces of C. Each open face is the interior of a face of C (in the linear subspace generated by this face).

Lemma 6.6 *Let $\varphi \in \Gamma(X, \mathrm{ad}(P))$ be a section regarded as a K-equivariant map $P \to \mathfrak{k}$. Suppose that the map*

$$B \xrightarrow{\ [\varphi]\ } \mathfrak{k}/K \xrightarrow{\ \simeq\ } C$$

takes values in a fixed open face c of C.

Then:

1. *The space $\varphi^{-1}(c)/T \subset P/T$ is a locally trivial fibre bundle with standard fibre $Z_K(c)/T$ over B.*

2. *Let A be any connection on P. The map $B \xrightarrow{\ [\varphi]\ } \mathfrak{k}/K$ is locally constant if and only if the projection of $d_{A,x}(\varphi)$ on $z_{\mathfrak{k}_x}(\varphi_x) \subset \mathfrak{k}_x := P_x \times_{\mathrm{ad}} \mathfrak{k}$ vanishes for every $x \in B$.*

Proof: 1. Note first that centralizer $Z_K(\gamma)$ of any element $\gamma \in c$ is the same, hence $Z_K(\gamma) = Z_K(c)$ for every $\gamma \in c$. Indeed, since K is connected, any centralizer of the form $Z_K(u)$ with $u \in \mathfrak{k}$ is connected, so it suffices to look at the Lie algebras of these centralizers. But the Lie algebra $z_k(u)$ has the following simple description in terms of the root set $R \subset \mathrm{Hom}_{\mathbb{R}}(\mathfrak{t}, i\mathbb{R})$:

$$z_k(u) = \left[\bigoplus_{\alpha \in R, \ \alpha(u)=0} \mathfrak{k}_{\alpha}^{\mathbb{C}} \right] \cap \mathfrak{k} \ .$$

Therefore it suffices to note that the map

$$\gamma \mapsto \{\alpha \in R | \ \alpha(\gamma) = 0\} \subset R$$

is constant on the open face c.

The map

$$ {}^{K}\!/_{Z_K(c)} \times c \longrightarrow \mathfrak{k} \ , \quad ([k], \gamma) \mapsto \mathrm{ad}_k(\gamma)$$

is an embedding, so its image $\mathrm{ad}_K(c)$ is a submanifold of $\mathfrak{k}$.

The restriction of the map

$$\varphi : P \to \mathrm{ad}_K(c)$$

to any fibre P_x is obviously a submersion, hence φ is also a submersion. The manifold $\varphi^{-1}(c)$ is obviously a principal $Z_K(c)$-bundle over B. Therefore, the quotient $\varphi^{-1}(c)/T$ is just the associated bundle with fibre $Z_K(c)/T$.

2. Let $s \in \mathrm{ad}_K(c)$. The Lie algebra $\mathfrak{k}$ decomposes as

$$\mathfrak{k} = z_{\mathfrak{k}}(s) \oplus [\mathfrak{k}, s] \ , \tag{82}$$

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whereas the tangent space at s of the submanifold $\mathrm{ad}_K(c) \subset \mathfrak{k}$ decomposes as

$$T_s(\mathrm{ad}_K(c)) = [\mathfrak{k}, s] \oplus [z_\mathfrak{k}(s) \cap T_s(\mathrm{ad}_K(c))] = T_s(\mathrm{ad}_k(s)) \oplus [z_\mathfrak{k}(s) \cap T_s(\mathrm{ad}_K(c))] \, .$$

Therefore, a map $\sigma : U \to c$ on a manifold U defines a locally constant map $[\sigma] : U \to \mathfrak{k}/K$ if and only if $\mathrm{pr}_{z_\mathfrak{k}(\sigma_x)}(d_x\sigma) = 0$ for all $x \in U$.

In our case, we deduce that the map $P \to \mathfrak{k}/K$ defined by $\varphi : P \to \mathfrak{k}$ is locally constant if and only if $\mathrm{pr}_{z_\mathfrak{k}(\varphi_p)}(d_p\varphi) = 0$ for all $p \in P$. But, for a vertical tangent vector $v \in T_p(P)$ one obviously has $d_p\varphi(v) \in [\mathfrak{k}, \varphi_x]$, so the condition $\mathrm{pr}_{z_\mathfrak{k}(\varphi_p)}(d_p\varphi) = 0$ is in fact equivalent with the condition $\mathrm{pr}_{z_{\mathfrak{k}_x}(\varphi_x)}(d_{A,x}\varphi) = 0$.
∎

Lemma 6.7 *Let K be a maximal compact subgroup of the reductive group G. The natural map*

$$\left. {}^{i\mathfrak{k}} \middle/ {}_K \right. \longrightarrow \left. {}^{\mathfrak{g}} \middle/\!\!\middle/ {}_G \right.$$

is injective.

Here we denoted by $\mathfrak{g} /\!\!/ G$ the GIT quotient of $\mathfrak{g}$ with respect to the adjoint action of G. In other words,

$$\mathfrak{g} /\!\!/ G = \mathrm{Spec}\{\mathbb{C}[\mathfrak{g}]^G\} \, .$$

Proof: The moment map for the K-action on $\mathfrak{g}$ is

$$m : \gamma \mapsto -\frac{i}{2}[\gamma, \gamma^*] \, ,$$

where $(\cdot)^*$ is the conjugation with respect to the real structure $\mathfrak{g} = [i\mathfrak{k}] \otimes \mathbb{C}$.

It suffices to note that $\mathfrak{k}$ is contained in $m^{-1}(0)$. ∎

Corollary 6.8 *Let $\iota_1, \ldots, \iota_k \in \mathbb{C}[\mathfrak{g}]$ be homogeneous generators of the invariant algebra $\mathbb{C}[\mathfrak{g}]^G$. Then two elements $u, v \in i\mathfrak{k}$ are in the same K-orbit if and only if $\iota_i(u) = \iota_i(v)$ for $1 \leq i \leq k$.*

In particular, a section σ in the adjoint bundle $P \times_{\mathrm{ad}} i\mathfrak{k}$ of a principal K-bundle is constant (respectively constant almost everywhere) if and only if the complex valued functions $\iota_i(\sigma)$ are constant (respectively constant almost everywhere) for $1 \leq i \leq k$.

The second statement in Lemma 6.6 holds for general possibly non-connected compact Lie groups. The converse statement is also true. More precisely, one has

Proposition 6.9 *Let K be a compact Lie group, P a principal K-bundle, A a connection on P and $\varphi \in \Gamma(X, \mathrm{ad}(P))$ regarded as a K-equivariant map $P \to \mathfrak{k}$.*

The map $B \xrightarrow{[\varphi]} \mathfrak{k}/K$ defined by φ is locally constant if and only if the projection $\mathrm{pr}_{z_{\mathfrak{k}_x}(\varphi_x)}(d_{A,x}\varphi)$ vanishes for every $x \in B$.

Proof:

We suppose that the base B is connected.

1. *The implication:*

$$[\varphi] \text{ is constant } \Rightarrow [\mathrm{pr}_{z_{\mathfrak{k}_x}(\varphi_x)}(d_{A,x}\varphi) = 0 \ \forall x \in B] \ .$$

In the case when K is connected, one just takes the open chamber c which contains the conjugacy class of $[\varphi]$ and applies Lemma 6.6.

The general case can be reduced to the connected case by taking a finite covering of the base B on which P admits a K_e-reduction $P_e \subset P$. It suffices to note that, in general, a continuous map $U \longrightarrow \mathfrak{k}/K_e$ on a connected manifold U is constant if and only if the induced map $U \longrightarrow \mathfrak{k}/K$ is constant.

2. *The implication:*

$$[\mathrm{pr}_{z_{\mathfrak{k}_x}(\varphi_x)}(d_{A,x}\varphi) = 0 \ \forall x \in B] \Rightarrow [\varphi] \text{ is constant} \ .$$

Let $j_i : \mathfrak{g}^{d_i} \to \mathbb{C}$ be the ad_G-invariant symmetric multilinear map which corresponds to ι_i (see Corollary 6.8).

One has
$$d[j_i(\varphi, \ldots, \varphi)] = d_i j_i(d_A\varphi, \varphi, \ldots, \varphi) \ .$$

On the other hand, taking into account that j_i is ad_G-invariant, it follows easily that, for any $v \in \mathfrak{g}$, the linear functional $j_i(\cdot, v, \ldots, v)$ vanishes on $[v, \mathfrak{k}]$. The point is that, for $v \in \mathfrak{k}$, one has an orthogonal decomposition

$$\mathfrak{k} = [v, \mathfrak{k}] \oplus z_{\mathfrak{k}}(v) \ .$$

Therefore, one gets

$$d[j_i(\varphi, \ldots, \varphi)] = d_i j_i(d_A\varphi, \varphi, \ldots, \varphi) = d_i j_i(\mathrm{pr}_{z_{\mathfrak{k}}(v)}(d_A(\varphi)), \varphi, \ldots, \varphi) = 0 \ .$$

The claim follows now from Corollary 6.8. ∎

In section 1.2 we have used the following technical result:

Proposition 6.10 *Let K be a maximal compact subgroup of a complex reductive group G and $s, \dot{s} \in i\mathfrak{k}$. Using the decomposition (82), write*

$$\dot{s} = \lambda + [k, s] \ ,$$

with $\lambda \in iz_{\mathfrak{k}}(s) \ , \ k \in z_{\mathfrak{k}}(s)^{\perp}$. Then

1.
$$(d_s \exp)(\dot{s})e^{-s} = \lambda + (k - \mathrm{ad}_{e^s} k) \ .$$

2.
$$\langle \dot{s}, p_{i\mathfrak{k}} \{(d_s \exp)(\dot{s})e^{-s}\} \rangle \geq \|\dot{s}\|^2 \ .$$

3.
$$\left\langle \left[p_{\mathfrak{k}} \{(d_s \exp)(\dot{s})e^{-s}\}, s\right], p_{i\mathfrak{k}} \{(d_s \exp)(\dot{s})e^{-s}\}\right\rangle \leq 0$$

with equality if and only if $\dot{s} \in iz_{\mathfrak{k}}(s)$.

Proof:

Put

$$\dot{s} = \lambda + [k, s] = \lambda + \frac{d}{dt}(\mathrm{ad}_{e^{tk}}(s)) \ , \ \lambda \in iz_{\mathfrak{k}}(s) \ , \ k \in z_{\mathfrak{k}}(s)^{\perp} \ .$$

One has

$$(d_s \exp)(\dot{s}) = (d_s \exp)(\lambda) + (d_s \exp)([k, s]) = \lambda e^s + \frac{d}{dt}|_{t=0} \, \mathrm{Ad}_{e^{tk}} \ e^s =$$

$$[\lambda + (k - \mathrm{ad}_{e^s} k)]e^s \ ,$$

so

$$\sigma := (d_s \exp)(\dot{s})e^{-s} = \lambda + (k - \mathrm{ad}_{e^s} k) \ .$$

Let R_s denote the set of eigenvalues of the endomorphism $[s, \cdot]$ on $\mathfrak{g}$. Decompose

$$k = \sum_{\rho \in R_s \backslash \{0\}} k_\rho \ , \ k_\rho \in \mathfrak{g}_\rho \ ,$$

with $\bar{k}_\rho = -k_{-\rho}$ (since $k \in \mathfrak{k}$). Then

$$[s, k] = \sum_{\rho \in R_s \backslash \{0\}} \rho \, k_\rho \ , \ \mathrm{ad}_{e^s} k = \sum_{\rho \in R_s \backslash \{0\}} e^\rho k_\rho \ ,$$

$$\dot{s} = \lambda - \sum_{\rho \in R_s \backslash \{0\}} \rho k_\rho \ , \ \sigma = \lambda + \sum_{\rho \in R_s \backslash \{0\}} (1 - e^\rho) k_\rho \ . \tag{83}$$

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We get

$$p_{i\mathfrak{k}}(\mathrm{ad}_{e^s}k) = \frac{1}{2}(\mathrm{ad}_{e^s}k + \overline{\mathrm{ad}_{e^s}k}) = \frac{1}{2}\sum_{\rho\in R_s\setminus\{0\}}(e^\rho - e^{-\rho})k_\rho \ ,$$

$$p_{\mathfrak{k}}(\mathrm{ad}_{e^s}k) = \frac{1}{2}(\mathrm{ad}_{e^s}k - \overline{\mathrm{ad}_{e^s}k}) = \frac{1}{2}\sum_{\rho\in R_s\setminus\{0\}}(e^\rho + e^{-\rho})k_\rho \ ,$$

$$p_{\mathfrak{k}}(k - \mathrm{ad}_{e^s}k) = \frac{1}{2}\sum_{\rho\in R_s\setminus\{0\}}(2 - e^\rho - e^{-\rho})k_\rho \ ,$$

$$[p_{\mathfrak{k}}(\sigma), s] = \frac{1}{2}\sum_{\rho\in R_s\setminus\{0\}}\rho(e^\rho + e^{-\rho} - 2)k_\rho = \frac{1}{2}\sum_{\rho\in R_s\setminus\{0\}}\rho e^{-\rho}(e^\rho - 1)^2 k_\rho \ ,$$

$$p_{i\mathfrak{k}}(\sigma) = \lambda + \frac{1}{2}\sum_{\rho\in R_s\setminus\{0\}}(-e^\rho + e^{-\rho})k_\rho \ .$$

Therefore,

$$\langle \dot{s}, p_{i\mathfrak{k}}(\sigma)\rangle = \|\lambda\|^2 + \frac{1}{2}\sum_{\rho\in R_s\setminus\{0\}}\rho(e^\rho - e^{-\rho})\|k_\rho\|^2 \geq$$

$$\geq \|\lambda\|^2 + \sum_{\rho\in R_s\setminus\{0\}}\rho^2\|k_\rho\|^2 = \|\dot{s}\|^2 \ ,$$

because the inequality $|e^x - e^{-x}| \geq 2|x|$ holds on $\mathbb{R}$.

For the third inequality we compute

$$\langle [p_{\mathfrak{k}}(\sigma), s], p_{i\mathfrak{k}}(\sigma)\rangle =$$

$$= -\sum_{\rho\in R_s\setminus\{0\}}\frac{1}{4}\rho(e^\rho - e^{-\rho})e^{-\rho}(e^\rho - 1)^2\|k_\rho\|^2 \leq 0 \ . \tag{84}$$

Equality occurs s if and only if $\sum_\rho \rho\, k_\rho = 0$. ■

6.3 Local maximal torus reductions of a K-bundle

We quote from [LT] the following Lemma (see [LT], section 7.4)

Lemma 6.11 *Let X be a manifold, and*

$$X = G_r \supset G_{r-1} \supset \ldots \supset G_1$$

a filtration by closed subsets. Define $F_k := G_k \setminus G_{k-1}$, $k = 2, \ldots, r$. Then

$$W := \bigcup_{k=2}^{r} \mathring{F}_k$$

is open and dense in X.

Proposition 6.12 *For any $f \in \Gamma(B, \mathrm{ad}(P))$ there is an open dense subset $W \subset X$ such that for every $x \in W$ there exists an open face $c_x \subset C$, an open neighborhood U_x of x in B and a T-reduction $\Pi_x \subset P|_{U_x}$ of the restriction $P|_{U_x}$ such that $f|_{U_x}$ is defined by a smooth map*

$$\lambda \in \mathcal{C}^\infty(U_x, c_x) \subset \mathcal{C}^\infty(U_x, \mathfrak{t}) = \Gamma(U_x, \mathrm{ad}(\Pi_x)) \ .$$

Proof:

Regard f as a K-equivariant map $P \to \mathfrak{k}$, and let $[f] : B \to \mathfrak{k}/K = C$ be the induced $\mathfrak{k}/K$-valued map on B. Consider the skeleton filtration

$$C = C_d \supset C_{d-1} \supset \ldots C_1 \supset \ldots \supset C_0$$

of C, where C_i is the union of all open faces of dimension less or equal i. We get a filtration by closed sets

$$B = B_d \supset B_{d-1} \supset \ldots B_1 \supset \ldots \supset B_0$$

of B, where $B_i := [f]^{-1}(C_i)$.

Consider now the locally closed subsets

$$F_i := B_i \setminus B_{i-1} = [f]^{-1}(C_i \setminus C_{i-1}) = [f]^{-1} \left[\coprod_{\substack{c \subset C \text{ open face} \\ \dim(c)=i}} c \right] \ .$$

By Lemma 6.11, it follows that $W := \bigcup \mathring{F}_i$ is dense in B. For any $x \in W$ let $i_x \in \{0, \ldots, d\}$ such that $x \in \mathring{F}_{i_x}$ and let U_x be an arbitrary contractible open neighborhood of x in $\mathring{F}_{i_x}$. Since U_x is connected, $[f](U_x)$ is contained in a single open face c_x of C. Moreover, the fibre bundle $\left[f|_{U_x}^{-1}(c_x) \right] / T$ over the contractible manifold U_x is trivial, hence one can find a section σ_x in this bundle. Regarding σ_x as a section in $[P|_{U_x}] / T$, we get a T-reduction of $P|_{U_x}$ on which f is given by a smooth c_x-valued function. ∎

Remark 6.13 *Proposition 6.12 is also true for non-connected Lie groups K. The proof reduces to the connected case using a finite cover of B, on which P can be reduced to the connected component $K_e \subset K$.*

6.4 Decomposing a connection with respect to a maximal torus reduction. Applications

Let P be a principal K-bundle over a manifold B and let $\iota : \Pi \hookrightarrow P$ be a T-reduction of P. Consider the root set $R \subset \mathrm{Hom}(\mathfrak{t}, i\mathbb{R})$ and the root decomposition

$$\mathfrak{g} = \bigoplus_{\rho \in R} \mathfrak{g}_\rho$$

of $\mathfrak{g} := \mathfrak{k}^{\mathbb{C}}$.

Since this decomposition is T-invariant, and we have fixed a T-reduction of P, one gets a decomposition of the complexified adjoint bundle $\mathrm{ad}^{\mathbb{C}}(P)$:

$$\mathrm{ad}^{\mathbb{C}}(P) = \bigoplus_{\rho \in R} \Pi \times_{\mathrm{ad}} \mathfrak{g}_\rho. \tag{85}$$

Denote by $\mathfrak{t}^\perp$ the natural complement of the Lie algebra $\mathfrak{t}$ in $\mathfrak{k}$, i. e.

$$\mathfrak{t}^\perp = \left[\bigoplus_{\rho \neq 0} \mathfrak{k}_\rho^{\mathbb{C}} \right] \cap \mathfrak{k} .$$

Let A be a connection on P, and $\omega_A \in A^1(P, \mathfrak{k})$ its connection form. Let A_Π be the induced connection on Π, and $\iota_*(A_\Pi)$ the push-forward connection to P. Write

$$\omega_A = \omega_{\iota_*(A_\Pi)} + a ,$$

where $a \in A^1(\Pi \times_{\mathrm{ad}} \mathfrak{t}^\perp)$ is the second fundamental form of the subbundle Π with respect to the connection A.

Suppose now that the basis B is a complex manifold. Then one can write $a = a^{10} + a^{01}$, where each term decomposes as

$$a^{10} = \sum_{\rho \neq 0} a_\rho^{10} , \ a_\rho^{10} \in A^{10}(\Pi \times_{\mathrm{ad}} \mathfrak{g}_\rho) , \ a^{01} = \sum_{\rho \neq 0} a_\rho^{01} , \ a_\rho^{01} \in A^{01}(\Pi \times_{\mathrm{ad}} \mathfrak{g}_\rho).$$

With respect to the usual real structure $\mathfrak{g} = [i\mathfrak{k}]^{\mathbb{C}}$ of $\mathfrak{g}$, one has

$$\overline{a_\rho^{10}} = -a_{-\rho}^{01} .$$

Let $\varphi \in A^0(\mathrm{ad}(P))$ be a smooth section. Suppose now that, with respect to the reduction Π, the section φ is induced by a smooth map $\lambda \in \mathcal{C}^\infty(B, c)$ where c is a fixed open face of the Weyl chamber C. We know, by Proposition 6.12, that locally we can always come to this situation.

Via the decomposition (85), and taking into account that $[a, \lambda] = -[\lambda, a]$, we have the following formulae for $\partial_A \varphi$, $\bar{\partial}_A \varphi$:

$$\partial_A \varphi = \partial \lambda - \sum_{\rho \neq 0} \rho(\lambda) a_\rho^{10} \ , \ \bar{\partial}_A \varphi = \bar{\partial} \lambda - \sum_{\rho \neq 0} \rho(\lambda) a_\rho^{01} = \bar{\partial} \lambda + \sum_{\rho \neq 0} \rho(\lambda) \overline{a_\rho^{10}} \ . \quad (86)$$

We will also need a formula for $e^{-s} \partial_A(e^s)$, where $s \in A^0(i \mathrm{ad}(P))$. Regard s (and e^s) as an ad-equivariant (respectively Ad-equivariant) $\mathfrak{g}$-valued (respectively G-valued) function on P. Let be $x \in X$, $p \in \Pi_x$ and $v \in T_x(X)$ and $w \in T_p(\Pi)$ the A_Π - horizontal lift of v.

The A-horizontal component of w is

$$w^A = w - [\omega_A(w)]_p^{\#} = w - a(w)_p^{\#} \ .$$

One has

$$e^{-s(p)} d(e^s)(w^A) = e^{-s(p)} \left[d(e^s)(w) - \frac{d}{dt}|_{t=0} \left(e^{s(p \exp(ta(w)))} \right) \right] =$$

$$= e^{-s(p)} \left[d(e^s)(w) - \frac{d}{dt}|_{t=0} \left(e^{\mathrm{ad}(\exp(ta(w)))^{-1} s(p)} \right) \right] =$$

$$= e^{-s(p)} \left[d(e^s)(w) - \frac{d}{dt}|_{t=0} \left(\mathrm{Ad}(\exp(ta(w)))^{-1}(e^{s(p)}) \right) \right] =$$

$$= e^{-s(p)} \left[d(e^s)(w) - [-a(w)e^{s(p)} + e^{s(p)} a(w)] \right] \ . \quad (87)$$

Denote now by λ the restriction $\lambda := s|_\Pi$. We get

$$e^{-s(p)} d(e^s)(w^A) = d\lambda(w) + \sum_{\rho \neq 0} (e^{-\rho(\lambda)} - 1) a_\rho(w) \ ,$$

hence

$$e^{-s} d_A(e^s) = d\lambda + \sum_{\rho \neq 0} (e^{-\rho(\lambda)} - 1) a_\rho \ , \quad (88)$$

and similarly

$$d_A(e^s) e^{-s} = d\lambda + \sum_{\rho \neq 0} (1 - e^{\rho(\lambda)}) a_\rho \ . \quad (89)$$

In section 4.4.1 we need the following important technical results.

Let E be a Hermitian space and $f : \mathbb{R} \to \mathbb{R}$ a real function. Using the spectral decomposition of Hermitian matrices, one can extend f to a map (denoted by the same symbol to save on notations) $\mathrm{Herm}(E) \to \mathrm{Herm}(A)$ by putting

$$f(\sum_i \lambda_i \mathrm{pr}_{F_i}) := \sum f(\lambda_i) \mathrm{pr}_{F_i}$$

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for every orthogonal direct sum decomposition of E.

It is known (see for instance [Bh]) that

Proposition 6.14 *The extension* $\mathrm{Herm}(E) \to \mathrm{Herm}(E)$ *of a continuous function* $f : \mathbb{R} \to \mathbb{R}$ *is continuous.*

In general, this extension inherits the regularity properties of f.

Remark 6.15   *1. If* $|f| \le |g|$ *then*

$$\|f(h)(v)\| \le \|g(h)(v)\| \ \forall v \in E, \ \forall h \in \mathrm{Herm}(E) .$$

2. More generally, if on a subset $A \subset \mathbb{R}$ *we have* $|f|_A| \le |g|_A|$, *then*

$$\|f(h)(v)\| \le \|g(h)(v)\| \ \forall v \in E, \ \forall h \in \mathrm{Herm}(E) \ with \ \mathrm{Spec}(h) \subset A .$$

Indeed, writing $h = \sum_i \lambda_i \mathrm{pr}_{F_i}$ as above, one gets

$$\|f(h)(v)\| = \sqrt{\sum_i f(\lambda_i)^2 \|\mathrm{pr}_{F_i}(v)\|^2} \le \sqrt{\sum_i g(\lambda_i)^2 \|\mathrm{pr}_{F_i}(v)\|^2} = \|g(h)(v)\| .$$

Denote by $\eta : \mathbb{R} \to \mathbb{R}_{>0}$ the <u>real</u> <u>analytic</u> function defined by

$$\eta(t) = \begin{cases} \sqrt{\frac{1-e^{-t}}{t}} & \text{if} \quad t \ne 0 \\ 1 & \text{if} \quad t = 0 . \end{cases}$$

Proposition 6.16 *Let* $\mathcal{Q}$ *be a holomorphic principal* $K^{\mathbb{C}}$*-bundle on a complex manifold* X, $P \subset \mathcal{Q}$ *a* K*-reduction, denote by* A^0 *the corresponding Chern connection, and let* $d_0 = \bar{\partial} + \partial_0$ *be the associated covariant derivative. Let* $s \in A^0(P \times_{\mathrm{ad}} i\mathfrak{k})$. *Then one has*

$$\left(i\Lambda_g \bar{\partial}(e^{-s}\partial_0(e^s)), s \right) = \frac{1}{2}P(|s|^2) + |\eta([s,\cdot])(\partial_0(s))|^2 . \tag{90}$$

Proof: Let T be a maximal torus of K and $R \subset \mathrm{Hom}(i\mathfrak{t}, \mathbb{R})$ be the root set. By Proposition 6.12 it suffices to check the formula in the case when P has a T-reduction $\Pi \subset P$ and s is given by a smooth function $\lambda : X \to i\mathfrak{t}$. By formulae (86) and (88) one has

$$\partial_0 s = \partial\lambda - \sum_{\rho \in R \setminus \{0\}} \rho(\lambda) a_\rho^{10} , \quad e^{-s}\partial_0(e^s) = \partial\lambda + \sum_{\rho \in R \setminus \{0\}} (e^{-\rho(\lambda)} - 1)a_\rho^{10} . \tag{91}$$

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Therefore, using the Hermitian inner product:

$$\left(i\Lambda_g \bar{\partial}(e^{-s}\partial_0(e^s))), s\right) = i\Lambda_g \bar{\partial}\left(e^{-s} \circ \partial_0(e^s), s\right) + i\Lambda_g(e^{-s} \circ \partial_0(e^s), \partial_0 s) =$$

$$= i\Lambda_g \bar{\partial}\left(\partial\lambda, \lambda\right) + i\Lambda_g(\partial\lambda, \partial\lambda) - \sum_{\rho \in R\backslash\{0\}} \rho(\lambda)(e^{-\rho(\lambda)} - 1)i\Lambda_g(a_\rho^{10}, a_\rho^{10}) =$$

$$(P(\lambda), \lambda) - \sum_{\rho \in R\backslash\{0\}} \rho(\lambda)(e^{-\rho(\lambda)} - 1)i\Lambda_g(a_\rho^{10}, a_\rho^{10}) \ . \tag{92}$$

On the other hand,

$$P(|s|^2) = i\Lambda_g \bar{\partial}\partial(|\lambda|^2) = 2\left(P(\lambda), \lambda\right) - |\partial\lambda|^2) \leq 2(P(\lambda), \lambda) \ . \tag{93}$$

Now it suffices to note that

$$|\partial\lambda|^2 - \sum_{\rho \in R\backslash\{0\}} \rho(\lambda)(e^{-\rho(\lambda)} - 1)i\Lambda_g(a_\rho^{10}, a_\rho^{10}) = |\eta([s, \cdot])(\partial_0(s))|^2 \ .$$

$\blacksquare$

6.5 Analytic results

Lemma 6.17 *Let E, F be vector bundles with inner products on a measurable space X with finite measure. Let $(u_n)_n$ be a sequence of L^2-sections on E weakly convergent in L^2 to a section u, and let $(v_n)_n$ be a sequence of L^2-sections on F strongly convergent in L^2 to a section v. Suppose that v_n converges almost everywhere to v and that $(|v_n|)_n$ is uniformly bounded by a positive constant M. Then it holds:*

1. *For every L^2-section ψ in $(E \otimes F)^\vee$, the sequence $(v_n\psi)_n$ of sections in $E^\vee$ converges strongly in L^2 to $v\psi$.*

2. *The sequence $(u_n \otimes v_n)_n$ converges weakly in L^2 to $(u \otimes v)$.*

Proof: 1. This follows from the Lebesgue dominated convergence theorem, since the sequence of non-negative functions $(|v_n\psi - v\psi|^2)_n$ converges almost everywhere to 0 and is bounded almost everywhere by the integrable function $4M^2|\psi|^2$.

2. We have:

$$\langle u_n \otimes v_n, \psi \rangle_{L^2} - \langle u \otimes v, \psi \rangle_{L^2} = \langle u_n, v_n\psi \rangle_{L^2} - \langle u, v\psi \rangle_{L^2} =$$

$$= \langle u_n, v_n\psi - v\psi \rangle_{L^2} + \langle u_n - u, v\psi \rangle_{L^2} \ .$$

The second term converges to 0 since, by hypothesis, $u_n \rightharpoonup u$ weakly. The first term tends to 0 because

$$|\langle u_n, v_n\psi - v\psi\rangle_{L^2}| \leq \|u_n\|_{L^2}\|v_n\psi - v\psi\|_{L^2} ,$$

$(u_n)_n$ is bounded[4] in L^2 and $\|v_n\psi - v\psi\|_{L^2} \to 0$ by 1. $\blacksquare$

Lemma 6.18 *Let $D^{n_i} \subset \mathbb{R}^{n_i}$ be the standard n_i-dimensional disk, $i = 1, 2$, and let $\varphi \in L_1^p(D^{n_1} \times D^{n_2})$, $1 \leq p < \infty$. Then for almost all $y \in D^{n_2}$ the following holds:*

1. *$\varphi|_{D^{n_1} \times \{y\}}$ belongs to $L_1^p(D^{n_1})$, $\partial_x\varphi|_{D^{n_1} \times \{y\}}$ belongs to $L^p(D^{n_1})$.*

2. *$d(\varphi|_{D^{n_1} \times \{y\}}) = (\partial_x\varphi)|_{D^{n_1} \times \{y\}}$ in $L^p(D^{n_1})$.*

Proof: By Fubini's theorem $\varphi|_{D^{n_1} \times \{y\}}$ and $d\varphi|_{D^{n_1} \times \{y\}}$ belong to $L^p(D^{n_1})$ for almost all $y \in D^{n_2}$. In particular, $\partial_x\varphi|_{D^{n_1} \times \{y\}}$ belongs to $L^p(D^{n_1})$ for almost all $y \in D^{n_2}$, too.

Let φ_n be a sequence of smooth functions converging to φ in the L_1^p - topology as $n \to \infty$. Therefore

$$\int_{D^{n_1} \times D^{n_2}} |\varphi_n - \varphi|^p \to 0 , \qquad \int_{D^{n_1} \times D^{n_2}} |d\varphi_n - d\varphi|^p \to 0 .$$

In particular,

$$\int_{D^{n_1} \times D^{n_2}} |\partial_x\varphi_n - \partial_x\varphi|^p \to 0 .$$

By Fubini's theorem, it follows that the positive functions

$$y \mapsto \int_{D^{n_1} \times \{y\}} |\varphi_n - \varphi|^p , \ y \mapsto \int_{D^{n_1} \times \{y\}} |\partial_x\varphi_n - \partial_x\varphi|^p$$

converge to 0 with respect to the L^1 - norm, so (taking a subsequence if necessary) one can assume that they converge to 0 almost everywhere on D^{n_2}. Therefore, for almost every $y \in D^{n_2}$, one has

$$\varphi_n|_{D^{n_1} \times \{y\}} \xrightarrow{L^p} \varphi|_{D^{n_1} \times \{y\}} , \ (\partial_x\varphi_n)|_{D^{n_1} \times \{y\}} \xrightarrow{L^p} (\partial_x\varphi)|_{D^{n_1} \times \{y\}} .$$

But, since the φ_n are smooth, one has

$$(\partial_x\varphi_n)|_{D^{n_1} \times \{y\}} = \partial_x[\varphi_n|_{D^{n_1} \times \{y\}}] .$$

[4] Any weakly convergent sequence in a Hilbert space is bounded.

One the other hand,

$$\partial_x[\varphi_n|_{D^{n_1} \times \{y\}}] \longrightarrow \partial_x[\varphi|_{D^{n_1} \times \{y\}}] \text{ as distribution}$$

(because the distribution limit of $\varphi_n|_{D^{n_1} \times \{y\}}$ is $\varphi|_{D^{n_1} \times \{y\}}$). Therefore,

$$\varphi_n|_{D^{n_1} \times \{y\}} \xrightarrow{L^p} \varphi|_{D^{n_1} \times \{y\}} \ , \ \partial_x[\varphi_n|_{D^{n_1} \times \{y\}}] \xrightarrow{L^p} \partial_x[\varphi|_{D^{n_1} \times \{y\}}] =$$

$$= (\partial_x \varphi)|_{D^{n_1} \times \{y\}} \ ,$$

which shows that 1. and 2. hold for such y. ■

6.6 Meromorphic parabolic reductions defined by weakly holomorphic L_1^2 - sections

Let M be a complex manifold. We begin with the following definition:

Definition 6.19 *A* <u>*measurable*</u> *map* $F : B^n \to M$ *is called "holomorphic on almost all lines" if*

1. *$n = 1$, F coincides almost everywhere with a holomorphic map $B \to M$.*

2. *$n > 1$, and for any holomorphic parameterization $D \times \Delta \xrightarrow{\varphi} U \subset B^n$ with $D \subset \mathbb{C}$, $\Delta \subset \mathbb{C}^{n-1}$ the restriction $F \circ \varphi(\cdot, \zeta) : D \to M$ is weakly holomorphic in the sense of 1., for almost all $\zeta \in \Delta$.*

We recall the following fundamental results (see Shiffman [Sh1], Uhlenbeck-Yau [UY1], [UY2]):

Theorem 6.20 *Any measurable map $F : B^n \to M$ from the ball $B^n \subset \mathbb{C}^n$ into a projective algebraic manifold M, which is holomorphic on almost all lines, coincides almost everywhere with a meromorphic map.*

Theorem 6.20 is used combined with an important regularity result (see Uhlenbeck-Yau [UY1], [UY2]) which applies to the case $n = 1$.

In order to state this result we need a brief preparation:

Let Y be a compact manifold (possibly with boundary) and N a closed submanifold of $\mathbb{R}^m$. We put

$$L_1^2(Y, N) := \{\varphi \in L_1^2(Y, \mathbb{R}^m) | \ \varphi(y) \in N \text{ for almost every } y \in Y\} \ ,$$

and we endow this set with the topology induced from $L_1^2(Y, \mathbb{R}^m)$. The space $L_1^2(Y, N)$ obviously depends on the embedding $Y \hookrightarrow \mathbb{R}^m$. When Y is a surface, one has the following important density property (see Schoen-Uhlenbeck [SU]).

Proposition 6.21 *Suppose that Y is a compact surface with possibly empty $\mathcal{C}^1$-boundary. Then $\mathcal{C}^\infty(Y, N)$ is dense in $L_1^2(Y, N)$.*

This result has the following subtle consequence:

Corollary 6.22 *Suppose that Y is a compact surface with possibly empty $\mathcal{C}^1$-boundary and let $\varphi \in L_1^2(Y, M)$. Then $d\varphi(T_y(Y)) \subset T_{\varphi(y)}(M)$ for almost all $y \in Y$.*

Proof: By the density property Proposition 6.21, one can find a sequence $\varphi_j \in \mathcal{C}^\infty(Y, N)$ which converges in the $L_1^2(Y, \mathbb{R}^m)$ - topology to φ. There exists a subsequence $(\varphi_{j_k})_k$ such that $\varphi_{j_k} \to \varphi$ and $d\varphi_{j_k} \to d\varphi$ almost everywhere as $k \to \infty$. It suffices to note that $T(N)$ is closed in $N \times \mathbb{R}^m$. $\blacksquare$

Now let $N \subset \mathbb{R}^m$ be a compact complex manifold embedded as a real submanifold in $\mathbb{R}^m$, and let J_N be the corresponding almost complex structure. The inclusion $T_x(N) \hookrightarrow \mathbb{R}^m$ of the tangent space at $x \in Y$ induces an embedding

$$T_x^{10}(N) \oplus T_x^{01}(N) = T_x^{\mathbb{C}}(N) \hookrightarrow \mathbb{C}^m \ .$$

Therefore, one gets a filtration

$$0 \subset T^{10}(N) \subset T^{\mathbb{C}}(N) \subset N \times \mathbb{C}^m$$

of the trivial complex rank m bundle over N.

Let $p : N \times \mathbb{C}^m \to T^{10}(N)$ be any bundle projection which induces the standard projection

$$T^{\mathbb{C}}(N) \longrightarrow T^{10}(N) \ , \ v \mapsto \frac{1}{2}(v - iJ_N(v))$$

on $T^{\mathbb{C}}(N)$.

Remark 6.23 *In the special case when the induced Riemannian metric on the submanifold $N \subset \mathbb{R}^m$ is Hermitian, one can just take the orthogonal projection with respect to the standard Hermitian structure on $N \times \mathbb{C}^m$ because, in this case, the direct sum $T^{10}(N) \oplus T^{01}(N)$ will be orthogonal with respect to this standard Hermitian structure.*

We can now state

Theorem 6.24 *[UY1], [UY2] Let $B \subset \mathbb{C}$ be the standard ball and let N be a compact Kähler manifold embedded as a real submanifold in $\mathbb{R}^N$. Let $f : B \to N$ be a map with the following properties:*

1. $f \in L_1^2(B, N)$.

2. f is weakly holomorphic, i. e. one of the following two equivalent conditions holds:

    (a) For almost all $b \in B$ the tangent map $T_b(B) \to T_{f(b)}(N)$ is $\mathbb{C}$-linear.

    (b) For almost all $b \in B$ it holds $p_{f(b)} \circ (\bar{\partial}_b f) = 0$, where $\bar{\partial} f$ stands for the $\bar{\partial}$-derivative $T_b^{01}(B) \to \mathbb{C}^m$ of $f : B \to \mathbb{C}^m$.

Then f coincides almost everywhere with a holomorphic map.

Combining Theorem 6.20 with the regularity Theorem 6.24, one gets the following important

Theorem 6.25 *Let $M \subset \mathbb{R}^m$ be a complex projective algebraic manifold embedded as a real submanifold of $\mathbb{R}^m$, and let $p : M \times \mathbb{C}^m \to T^{10}(M)$ be a bundle projection which agrees with the standard projection $T^{\mathbb{C}}(M) \to T^{10}(M)$ on the subbundle $T^{\mathbb{C}}(M) \subset M \times \mathbb{C}^m$. Then any map $f \in L_1^2(B^n, M)$ satisfying the equation*

$$p \circ \bar{\partial} f = 0$$

almost everywhere on B^n, coincides almost everywhere with a meromorphic map.

Proof: This follows easily from the Theorem 6.20, the regularity Theorem 6.24 and Lemma 6.18. ∎

Let K be a maximal compact subgroup of the reductive group G. We will apply these results to an algebraic manifold of the form

$$M = {}^G\!/_{G(s_0)} \,,$$

where $G(s_0)$ is the parabolic subgroup defined by an element $s_0 \in i\mathfrak{k}$. We will need the following simple result.

Proposition 6.26 *i) The map*

$$\mathrm{ad}_K(s_0) = {}^K\!/_{Z_K(s_0)} \to {}^G\!/_{G(s_0)}$$

is a diffeomorphism.

ii) Endow $i\mathfrak{k}$ with an ad_K-invariant inner product and consider the embedding $M = \mathrm{ad}_K(s_0) \hookrightarrow i\mathfrak{k}$ of the projective algebraic manifold M in the Euclidean space $i\mathfrak{k}$ defined by the identification $M = \mathrm{ad}_K(s_0)$ in i). Then

1. For any $s \in \mathrm{ad}_K(s_0) = M$, the images of the morphisms

$$T_s^{10}(M) \to (i\mathfrak{k}) \otimes \mathbb{C} = \mathfrak{g}, \ T_s^{01}(M) \to (i\mathfrak{k}) \otimes \mathbb{C} = \mathfrak{g}$$

induced by this embedding are $\mathfrak{g}_s^\pm$, where

$$\mathfrak{g}_s^\pm := \bigoplus_{\pm\lambda>0} \mathrm{Eig}([s,\cdot],\lambda) \ .$$

2. The induced Riemannian metric on M is Kähler and homogeneous.

Proof: i) We just have to prove that the natural map

$$\left. K \middle/ Z_K(s_0) \right. \longrightarrow \left. G \middle/ G(s_0) \right.$$

is bijective. For surjectivity, let $g \in G$. We can decompose g as $g = ku$, where $u \in \exp(i\mathfrak{k})$. Let $B \subset G(s_0) \cap G_e$ be a Borel subgroup contained in the parabolic subgroup $G(s_0) \cap G_e$ of the connected reductive group G_e (the connected component of e in G). Since $\exp(i\mathfrak{k}) \in G_e$, one can further decompose $u = lb$, where $l \in K_e$ and $b \in B$, so one gets $g = (kl)b$, which shows that $g \equiv kl$ mod $G(s_0)$.

For injectivity, it suffices to show that

$$K \cap G(s_0) = Z_K(s_0) \ . \tag{94}$$

The inclusion $Z_K(s_0) \subset K \cap G(s_0)$ is obvious. The opposite inclusion follows by choosing an embedding $G \hookrightarrow GL(n,\mathbb{C})$ which maps K into $U(n)$. Let Φ be the filtration of $\mathbb{C}^n$ defined by the eigenspaces of the Hermitian matrix s_0. More precisely, the k-term of this filtration is the direct sum of the eigenspaces corresponding to the first k eigenvalues of s_0. It is well known that the parabolic subgroup $GL(n,\mathbb{C})(s_0) \subset GL(n,\mathbb{C})$ is just the stabilizer of the filtration Φ. On the other hand, one has $G(s_0) = G \cap GL(n,\mathbb{C})(s_0)$. Let $g \in K \cap G(s_0)$. The matrix defined by g is unitary and leaves invariant the filtration Φ, so it leaves invariant all the eigenspaces of s_0 in $\mathbb{C}^n$. Therefore g commutes with s_0.

ii)

1. Let $s = \mathrm{ad}_k(s_0) \in \mathrm{ad}_K(s_0)$. The tangent space $T_s(\mathrm{ad}_K(s_0))$ is just the subspace $\mathrm{ad}_k([\mathfrak{k}, s_0]) = [\mathfrak{k}, s] \subset i\mathfrak{k}$. The point which corresponds to s via our identification $\mathrm{ad}_K(s_0) = G/G(s_0)$ is the congruence class $c := kG(s_0) \in G/G(s_0)$. The tangent space of $G/G(s_0)$ at c is

$$T_c(G/G(s_0)) = \left. (L_k)_*(\mathfrak{g}) \middle/ (L_k)_*(\mathfrak{g}(s_0)) \right. \ .$$

Now note that $\mathfrak{g}_{s_0}^+$ is a complement of $\mathfrak{g}(s_0)$ in $\mathfrak{g}$, so one gets a natural isomorphism $T_c(G/G(s_0)) = (L_k)_*(\mathfrak{g}_{s_0}^+) = (R_k)_*(\mathfrak{g}_s^+)$.

The isomorphism $T_s(\mathrm{ad}_K(s_0)) \simeq T_c(G/G(s_0))$ induced by our identification is given by

$$[\mathfrak{k}, s] \longrightarrow \mathfrak{g}_s^+ \xrightarrow{(R_k)_*} (R_k)_*(\mathfrak{g}_s^+) \ ,$$

where the first arrow is just the restriction $\rho_s : [\mathfrak{k}, s] \to \mathfrak{g}_s^+$ of the projection $i\mathfrak{k} \to \mathfrak{g}_s^+$. The complex structure on the tangent space $[\mathfrak{k}, s]$ is the pull-back of the natural complex structure of $\mathfrak{g}_s^+$ via ρ_s. Therefore, the subspaces $T_s^{10}(\mathrm{ad}_K(s_0))$, $T_s^{01}(\mathrm{ad}_K(s_0))$ of the complexified tangent space $T_s^{\mathbb{C}}(\mathrm{ad}_K(s_0)) = [\mathfrak{g}, s] = \mathfrak{g}_s^+ \oplus \mathfrak{g}_s^-$ are just $\mathfrak{g}_s^+$, $\mathfrak{g}_s^-$.

2. It is easy to see that the Riemannian metric induced by the embedding $\mathrm{ad}_K(s_0)$ is Hermitian and homogeneous, i. e. invariant under the natural K-action. The fact that the corresponding Kähler metric ω is closed follows from the identity

$$L_X\omega = d\ \iota_X\omega + \iota_X d\omega$$

applied to the fundamental vector fields associated with the transitive K-action on $\mathrm{ad}_K(s_0)$.

$\blacksquare$

Applying Theorem 6.25, we get

Proposition 6.27 *Let* $\varphi \in L_1^2(B^n, i\mathfrak{k})$ *such that* $\varphi(x) \in \mathrm{ad}_K(s_0)$ *for almost every* $x \in B^n$. *Suppose that*

$$\mathrm{pr}_{\mathfrak{g}_{\varphi(x)}^+}(\bar{\partial}_x\varphi) = 0$$

for almost every $x \in B^n$. *Then* φ *coincides almost everywhere with a meromorphic map* $B^n \to M = \mathrm{ad}_K(s_0)$.

More generally, let $\mathcal{Q}$ *be a holomorphic G-bundle on a complex manifold X, let $P \subset \mathcal{Q}$ be a K-reduction, and let* $\sigma \in L_1^2(P \times_{\mathrm{ad}} i\mathfrak{k})$ *be an L_1^2 section which defines an almost everywhere constant map* $[\sigma] : X \to i\mathfrak{k}/K$ *such that*

$$\mathrm{pr}_{V_{\sigma(x)}^+}(\bar{\partial}_x\sigma) = 0 \tag{95}$$

for almost every $x \in X$. *Fix a representative σ_0 in the conjugacy class defined by σ. Then σ defines a meromorphic reduction of $\mathcal{Q}$ to $G(\sigma_0)$.*

In the regular case, the holomorphic map $B^n \to G/G(s_0)$ associated with a smooth map $\varphi : B^n \to \mathrm{ad}_K(s_0)$ satisfying the holomorphy condition

$$\mathrm{pr}_{V_{\sigma(x)}^+}(\bar{\partial}_x\sigma) = 0 \ \forall x \in B^n$$

is given explicitly by the formulae

$$x \mapsto \text{the class mod } G(s_0) \text{ of an element } k \in K \text{ with } \mathrm{ad}_k(s_0) = \varphi(x) \ ,$$

$$x \mapsto \{g \in G| \ \mathrm{ad}_{g^{-1}}(\varphi(x)) \in \mathfrak{g}(s_0)\} \ .$$

Similarly, the $G(s_0)$-reduction defined by a regular section $\sigma \in A^0(P \times_{\mathrm{ad}} i\mathfrak{k})$ satisfying our weak holomorphy condition is

$$\mathcal{Q}(\sigma) = \{q \in \mathcal{Q}| \ \sigma(q) \in \mathfrak{g}(\sigma_0)\} \ .$$

The converse of Proposition 6.27 is also true; more precisely

Proposition 6.28 *Let $\rho \subset \mathcal{Q}/G(\sigma_0)$ be a meromorphic $G(\sigma_0)$-reduction of a holomorphic principal G-bundle $\mathcal{Q}$ on a compact complex manifold X, and let $P \subset \mathcal{Q}$ be any K-reduction.*

Then the section σ associated with σ_0 via the $K \cap G(\sigma_0) = Z_K(\sigma_0)$-reduction $P \cap \mathcal{Q}^\rho$ of $P|_{X_\rho}$ defines an element in $L^2_1(P \times_{\mathrm{ad}} i\mathfrak{k})$ which satisfies the weak holomorphy condition (95) and whose associated meromorphic $G(\sigma_0)$-reduction is ρ.

Proof: The crucial point here is the fact that $\sigma \in L^2_1(P \times_{\mathrm{ad}} i\mathfrak{k})$. Let A be the Chern connection of P. We make use of the formula (22) which was obtained in the proof of Theorem 3.3.

$$\int_{X_\rho} \langle i\Lambda_g F_A, \sigma)\rangle \, vol_g = \frac{2\pi}{(n-1)!} \deg(\rho, h(\xi)) - \sum_{\lambda \in \mathrm{Spec}^+[\xi,\cdot]} \lambda \parallel a_\lambda \parallel^2_{L^2} \ . \qquad (96)$$

On the other hand, by formula (61), one has on X_ρ

$$\partial_0(\sigma|_{X_\rho}) = \sum_{\lambda \in \mathrm{Spec}^+[\sigma_0,\cdot]} [a_\lambda, (\sigma|_{X_\rho})] = - \sum_{\lambda \in \mathrm{Spec}^+[\sigma_0,\cdot]} \lambda a_\lambda \ . \qquad (97)$$

Comparing these formulae, one gets immediately

$$\parallel \partial_0(\sigma|_{X_\rho}) \parallel_{L^2} < \infty \ .$$

On the other hand, one can easily check that the form $\partial_0(\sigma|_{X_\rho}))$, regarded as an L^2 - form on the whole manifold X, is in fact the distribution ∂_0 - derivative of the L^∞ - section σ. The proof uses the same argument as for the analogous result for orthogonal projections on subsheaves of Hermitian holomorphic bundles. ∎

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